Multi-party Purity Distillation and Instrument Simulation in the One-Shot Regime

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***Abstract*—We address the problem of distributed multi-party purity distillation (PD) involving three parties in the one-shot regime. By obtaining a one-shot inner bound for the distributed instrument simulation problem that naturally generalizes to the best known asymptotic inner bound, and combining it with a recent one-shot single-party local purity concentration protocol [1], we design a one-shot multi-party PD protocol, analyze its performance and derive a new inner bound. The derived inner bound naturally generalizes to the best known asymptotic inner bound [2].**

I. Introduction

Purity distillation is a fundamental task in quantum information theory, with deep connections to thermodynamics and resource-theoretic formulations of information processing. In its simplest form, it asks how much local purity can be distilled from a noisy quantum resource via local unitaries. When one considers a two-party scenario, wherein distributed parties Alice and Bob have access to copies of a mixed state ρ^{AB} and have to distil local pure states, an additional resource of one-way classical communication is necessary. Building on [3], [4] formulated a *purity distillation* (PD) problem that studies this trade-off. Specifically, [4] permitted the parties to (i) perform local unitaries, (ii) use pre-prepared ancillas, so long as they are returned. Studying the trade-off between the number R of bits that Alice must send Bob to distil a total of P pure qubits amidst them in a conventional asymptotic Shannon theoretic framework, [4] provided a single-letter characterization.

A broader look at the PD problem reveals connections to the complementary problem of entanglement distillation. Indeed, the obtained rates inform us of the amount of local and non-local information present in bipartite states. This motivates us to investigate the PD problem in a multi-party scenario in a more generalized Shannon theoretic framework. Specifically, we consider a scenario wherein three distributed parties Alice_1, Alice_2 and Charlie have access to a **single** copy of a tripartite state $\rho^{A_1A_2C}$. Classical communication between senders Alice_1, Alice_2 and receiver (Rx) Charlie is modeled as a multiple-access dephasing channel. How many bits R_i must Alice_i communicate to Charlie to distill a total of P pure states amidst the three parties? Our focus in this article is to characterize the set of all rate triples (R_1, R_2, P) for which there exists a PD protocol guaranteeing these rates.

A combination of two aspects central to our study distinguishes it from all prior work. Firstly, the distributed multi-party aspect involving three components of an entangled state entails elements not addressed in a two-party scenario. Indeed, entanglement between the A_1 and A_2-systems implies that an efficient protocol must remove the correlation between the bits sent by the senders - an aspect not addressed in two-party studies [1], [3], [4]. Secondly, we study this problem in the general one-shot regime. Crucially, this forces us to design protocols that cannot leverage techniques such as time sharing which can greatly simplify protocol design. We are thus forced to design one-shot protocols and analyze them without the benefit of concentration and tools such as typical subspaces. [2], as discussed later, undertake an asymptotic study.

Throughout, we focus on characterizing inner bounds. We design a one-shot purity distillation protocol for the multi-party setup and analyze its information theoretic performance. Our main contribution is a characterization of a one-shot inner bound (Thm. 3) for the multi-party purity distillation. The derived inner bound is optimal in the sense that its natural asymptotic generalization yields the current known best inner bound for the multi-party purity distillation in the asymptotic regime. As elaborated below, our pursuit also yields a one-shot inner bound (Thm. 1) for the problem of instrument simulation involving three distributed parties.

A key element in optimizing communication rates in a PD protocol is measurement compression (MC). Indeed, distilling pure states from mixed states with minimal bit rates forces the protocol to compress measurement outcomes. In fact, since the state acted upon needs to be preserved, MC protocols [5], [6] are insufficient since these only preserve the reference. We are thus led to formulate and solve a general problem of compressing a separable instrument acting on a two-component state in the one-shot regime. Characterizing a new inner bound (Thm. 1) to the instrument simulation (IS) problem, we then employ this in the multi-party PD. Employing the novel one-shot single-party purity concentration protocol of [7], we design our one-shot multi-party PD protocol. With regard to our analysis, we rely on Sen's one-shot fully smooth multi-party covering [8].

A survey of previous works allows us to highlight the new challenges we encounter and our tools. [3], [4] were focussed on the then popular asymptotic regime. Recently, the novel work of Chakraborty et al. [1], [7] has designed and analyzed PD protocols in the one-shot regime and obtained tight characterizations. As discussed earlier, efficient PD crucially relies on efficient measurement simulation, i.e. measurement compression (MC). The complexity of MC protocols had restricted MC studies to simulation of a single-component measurement. [9] developed a time-sharing protocol to compress separable

measurements acting on two components of a distributed state. Building on this, [2] has characterized inner bounds for the multi-party PD problem studied here. However, since they rely on time sharing and typicality based arguments, they are inapplicable in the one-shot regime.

Designing and analyzing a general distributed MC protocol that does not rely on time sharing and preserves the states it acts on is the main challenge we face. The two simulation measurements employed by the parties, while designed based on a single component measurement, must preserve the post-measurement entanglement. In [10], we identify a compatible operator sliding (COS) tool to accomplish this. However, [10] does not preserve the state acted upon. Moreover, the simulation protocol must preserve the acted upon state. Here, by elevating the COS tool, we accomplish efficient instrument simulation in the one-shot regime. In addition to being an independent contribution, this enables us to solve the multi-party PD in the one-shot regime. Owing to the page-limit constraints and the involved nature of a quantum analysis, we provide detailed proofs in [11].

II. Preliminaries and Problem Statements

Notation: We supplement notation in [12] with the following. For $n \in \mathbb{N}$, we let $[n] \triangleq \{1, \cdots, n\}, [\overline{n}] \triangleq \{0\} \cup [n]$. $\mathcal{H}_A$ and $\mathcal{H}_X$ will denote state spaces of the accessible and inaccessible (reference) components of our quantum systems respectively. Throughout, all Hilbert spaces are assumed to be finite-dimensional and sets to be of finite cardinality. An underline denotes an appropriate aggregation of related objects. For ex. if $\mathcal{H}_{A_1}, \mathcal{H}_{A_2}$ are Hilbert spaces, $\mathcal{H}_{\underline{A}}$ denotes $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, whereas when $k_j \in [K_j]$ for $j \in [2]$ are elements in index sets, $\underline{k} = (k_1, k_2)$ or we let $p_{Y|\underline{W}}$ denote $p_{Y|W_1W_2}$. $\mathcal{L}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ denote linear, positive and density operators acting on $\mathcal{H}$. For a stochastic matrix $(p_{Y|W}(y|w) : (w, y) \in \mathcal{W} \times \mathcal{Y})$, we let $\overline{\mathscr{E}}_p^{Y|W}(\cdot), \mathscr{E}_p^{Y|W}(\cdot)$ denote the CPTP maps $\overline{\mathscr{E}}_p^{Y|W}(a) \triangleq \sum_{(w,y)\in\mathcal{W}\times\mathcal{Y}} p_{Y|W}(y|w) \, |w\rangle \langle w|a|w\rangle\langle w| \otimes |y\rangle\langle y|$ and $\mathscr{E}_p^{Y|W}(a) \triangleq \sum_{(w,y)\in\mathcal{W}\times\mathcal{Y}} p_{Y|W}(y|w) \, \langle w|a|w\rangle \, |y\rangle\langle y|$. CP, TNI, TP abbreviate completely positive, trace non-increasing, trace preserving respectively. The definition of I_H^ϵ is recalled in Section 1.4 of [13], while the definition of I_2^ϵ appears in Definition 1 of [8].

Rem 1. *Our choice of notation is motivated by the intent to keep our presentation consistent with earlier works on PD [2], [4]. Owing to multi-party nature with 3 components, we admit the notation is slightly involved.*

A. Purity Distillation : Problem Statement

Distributed parties Alice$_1$, Alice$_2$ and Charlie hold respective components (systems) of a tripartite state $\rho^{A_1A_2C}$. The goal is that each party distills pure states by performing local unitary operations. To accomplish this, they share a multiple-access dephasing channel from (Alice$_1$,Alice$_2$)$\to$Charlie. In addition, each party may catalytically use pre-prepared pure ancilla qubits to "pump prime" their *distillation protocol*, which must however be returned at the end. Following these remarks on notation, we formalize a purity distillation protocol and quantify its ability by the total dimension of the distilled purity.

Rem 2. *Since we have three systems in PD, we let a single underline, as stated earlier, aggregate the first two components and a double underline aggregate all three components. For ex. $\underline{l}$ represents triple (l_1, l_2, l_3). As will be clear in the sequel, system C will correspond to index 3 and Alice$_i$ with index i for $i \in [2]$.*

Defn 1. *Given Hilbert spaces $(\mathcal{H}_{A_i} : i \in [2], \mathcal{H}_C)$, a tripartite purity distillation protocol $(\underline{M}, \underline{l}, \underline{s}, \mathcal{U}_{\underline{A}}, \mathcal{N}, \mathcal{U}_C) = (M_1, M_2, l_1, l_2, l_3, s_1, s_2, s_3, \mathcal{U}_{\underline{A}}, \mathcal{N}, \mathcal{U}_C)$ comprises*

- *Alice$_i$'s local unitary $\mathcal{U}_{A_i} : \mathcal{H}_{A_i} \otimes \mathcal{H}_{A_i^0} \to \mathcal{H}_{A_i^p} \otimes \mathcal{H}_{A_i^g} \otimes \mathcal{H}_{M_i}$ with $\dim(\mathcal{H}_{A_i^p}) = s_i$, $\dim(\mathcal{H}_{A_i^0}) = l_i$ and $\dim(\mathcal{H}_{M_i}) = M_i$ for $i \in [2]$,*
- *A multi-access dephasing channel $\mathcal{N} : \mathcal{L}(\mathcal{H}_{\underline{M}}) \to \mathcal{L}(\mathcal{H}_{\underline{M}})$, where $\mathcal{H}_{\underline{M}} = \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}$*
- *Charlie's local unitary $\mathcal{U}_C : \mathcal{H}_C \otimes \mathcal{H}_{C_0} \otimes \mathcal{H}_{\underline{M}} \to \mathcal{H}_{C_p} \otimes \mathcal{H}_{C_g}$ with $\dim(\mathcal{H}_{C_p}) = s_3$, $\dim(\mathcal{H}_{C_0}) = l_3$.*

Defn 2. *Given a tripartite quantum state $\rho^{A_1A_2C}$, an $\eta-$pure state can be distilled at rate P with communication cost (R_1, R_2) if there exists a tripartite purity distillation protocol $(\underline{M}, \underline{l}, \underline{s}, \mathcal{U}_{\underline{A}}, \mathcal{N}, \mathcal{U}_C)$ such that $\log M_i \le R_i$ for $i \in [2]$, $\sum_{i=1}^{3} \log s_i - \log l_i \ge P$ and*

$$\left\| \omega_{A_1^pA_2^pC_p} - |0\rangle\langle 0|_{A_1^p} \otimes |0\rangle\langle 0|_{A_2^p} \otimes |0\rangle\langle 0|_{C_p} \right\|_1 \le \eta.$$

where $\omega = \mathcal{U}_C \circ \mathcal{N} \circ \mathcal{U}_{A_1}\mathcal{U}_{A_2}(\rho_{A_1A_2C A_1^0A_2^0C_0})$ and $\rho_{A_1A_2CA_1^0A_2^0C_0} = \rho^{A_1A_2C} \otimes |0\rangle\langle 0|_{A_1^0} \otimes |0\rangle\langle 0|_{A_2^0} \otimes |0\rangle\langle 0|_{C_0}$. In this case, we say the triple (P, R_1, R_2) is $\eta-$purity-achievable.

We say pure states can be perfectly distilled at rate P with communication cost (R_1, R_2) if for every $\eta > 0$, there exists $N_\eta \in \mathbb{N}$ such that for all $n \ge N_\eta$, n copies of the tripartite state $\otimes_{t=1}^n \rho^{A_1A_2C}$, can yield $\eta-$pure states at rate nP with communication cost (nR_1, nR_2). In this case, we say triple (P, R_1, R_2) is perfectly purity achievable.

B. Instrument Simulation : Objects and Problem Statements

A *quantum instrument* $\mathcal{J} : \mathcal{L}(\mathcal{H}_A) \longrightarrow \mathcal{L}(\mathcal{H}_B \otimes \mathbb{C}^{|\mathcal{Y}|})$ with outcome alphabet $\mathcal{Y}$ is a collection $\{\mathcal{J}_y : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)\}_{y\in\mathcal{Y}}$ of CP TNI maps such that the total map $\sum_{y\in\mathcal{Y}} \mathcal{J}_y$ is trace preserving (TP). Equivalently,

$$\mathcal{J}(\rho) = \mathrm{Tr}_E \left[\sum_{y\in\mathcal{Y}} M_y \, \rho \, M_y^\dagger \otimes |y\rangle\langle y| \right],$$

with operators $M_y : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that $\sum_{y\in\mathcal{Y}} M_y^\dagger M_y = I_A$ and an environment $\mathcal{H}_E$ satisfying $\dim \mathcal{H}_A \le \dim \mathcal{H}_B \otimes \mathcal{H}_E$. In the latter formulation, the classical outcome y is coherently stored in an orthonormal basis $\{|y\rangle : y \in \mathcal{Y}\}$ of a classical register. Our focus will be on simulating a separable instrument defined below.

Defn 3. *An instrument $\mathcal{J} : \mathcal{L}(\mathcal{H}_{\underline{A}}) \longrightarrow \mathcal{L}(\mathcal{H}_{\underline{B}} \otimes \mathbb{C}^{\mathcal{Y}})$ acting[1] on a bipartite state specified through a collection*

[1]Recall our underline notation. $\mathcal{H}_{\underline{A}}$ denotes $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and so on.

$\{\mathcal{J}_y : \mathcal{L}(\mathcal{H}_{\underline{A}}) \to \mathcal{L}(\mathcal{H}_{\underline{B}}) : y \in \mathcal{Y}\}$ *of CP TNI maps is separable if there exists, for* $i \in [2]$ *instruments* $\mathcal{I}^{(i)} : \mathcal{L}(\mathcal{H}_{A_i}) \longrightarrow \mathcal{L}(\mathcal{H}_{B_i} \otimes \mathbb{C}^{\mathcal{W}_i})$ *specified through a collection* $\{\mathcal{I}^{(i)}_{w_i} : \mathcal{L}(\mathcal{H}_{A_i}) \to \mathcal{L}(\mathcal{H}_{B_i}) : w_i \in \mathcal{W}_i\}$ *of CP TNI maps and a stochastic matrix* $p_{Y|W_1W_2}$ *such that, for each* $y \in \mathcal{Y}$ *we have*

$$\mathcal{J}_y = \sum_{w_1,w_2} p_{Y|W_1W_2}(y|w_1,w_2)\mathcal{I}^{(1)}_{w_1} \otimes \mathcal{I}^{(2)}_{w_2}. \tag{1}$$

The scenario of interest involves three distributed parties - Alice$_1$, Alice$_2$ and Charlie. While Charlie holds classical register Y, for $i \in [2]$ Alice$_i$ holds system A_i of the bipartite state $\rho^{A_1A_2}$. The goal is to simulate the action of instrument

$$\mathcal{J} : \mathcal{L}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \longrightarrow \mathcal{L}(\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \mathbb{C}^{\mathcal{Y}}),$$

with finite outcome set $\mathcal{Y}$ such that the classical outcome Y available at Charlie and the post-instrument quantum states are available at Alice$_1$, Alice$_2$ respectively. To achieve this, each sender Alice$_i$: $i \in [2]$ shares with Charlie (i) a limited rate classical communication link modeled as dephasing channels, and (ii) statistically independent common randomness.

We first note that to simulate a separable instrument $\mathcal{J}$, it suffices for each party Alice$_i$ to apply instrument $\mathcal{I}^{(i)}$ and let Charlie stochastically process the outcomes w_1, w_2 through the stochastic matrix $p_{Y|W_1W_2}$. We henceforth focus exclusively on simulating $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$. The main idea now is that, instead of implementing the full instruments with outcome alphabets $\mathcal{W}_1$ and $\mathcal{W}_2$, each Alice$_i$ applies an encoding instrument with a much smaller number $M_i \ll |\mathcal{W}_i|$ of outcomes and sends the resulting classical message to Charlie. Using these messages together with the corresponding shared randomness, Charlie stochastically post-processes outcomes $\hat{W}_1 \in \mathcal{W}_1$ and $\hat{W}_2 \in \mathcal{W}_2$ to $\hat{Y} \in \mathcal{Y}$ using $p_{Y|W_1W_2}$. The resulting post-instrument quantum state, which remains in the possession of Alice$_1$ and Alice$_2$ in conjunction with the classical register Y at Charlie, is required to be indistinguishable from that obtained by directly applying the target instruments which is given by

$$\mathcal{T} \triangleq (\mathrm{id}_X \otimes \mathcal{J})\left\{\left|\varphi^{XA_1A_2}\right\rangle\!\!\left\langle\varphi^{XA_1A_2}\right|\right\}. \tag{2}$$

We therefore need a protocol to use the dephasing channels and common randomness. We formalize the same and precisely state the multi-party instrument simulation problems.

Defn 4. *A* $(\underline{K}, \underline{M}, \underline{\mathcal{E}}, \mathcal{D})$ *one-shot instrument simulation (IS) protocol consists of (i) a bank of* K_i *instruments* $\mathcal{E}^{(i)}_{k_i} : \mathcal{L}(\mathcal{H}_{A_i}) \longrightarrow \mathcal{L}(\mathcal{H}_{B_i} \otimes \mathbb{C}^{M_i}) : k_i \in [K_i]$ *each with* M_i *outcomes for* $i \in [2]$*, and a (ii) decoder instrument* $\mathcal{D}_{\underline{k}} : \mathcal{L}(\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}) \longrightarrow \mathcal{L}(\mathbb{C}^{|\mathcal{Y}|})$. ***$\mathcal{J}$'s action on $\rho^{A_1A_2}$ can be η-simulated with (communication) cost*** $(\underline{R}, \underline{C})$ *if there exists a* $(\underline{K}, \underline{M}, \underline{\mathcal{E}}, \mathcal{D})$ *one-shot IS protocol for which* $\log K_i \le C_i$, $\log M_i \le R_i$ *for* $i \in [2]$ *and state* $\mathcal{T}$ *in* (2) *satisfies*

$$\left\| \left(\mathrm{id}_X \otimes \sum_{k_1,k_2} \frac{\left[\mathcal{D}_{\underline{k}} \circ \{\mathcal{E}^{(1)}_{k_1} \otimes \mathcal{E}^{(2)}_{k_2}\}\right]}{K_1K_2} \right)(\phi_{\rho^{XA_1A_2}}) - \mathcal{T} \right\|_1 \le \eta. \tag{3}$$

$\mathcal{J}$'s action on $\rho^{A_1A_2}$ can be perfectly simulated with (communication) cost $(\underline{R}, \underline{C})$ *if for every* $\eta > 0$*, there exists* N_η *such that for all* $n \ge N_\eta$, $\otimes_{t=1}^n \mathcal{J}$*'s action on* $\otimes_{t=1}^n \rho^{A_1A_2}$ *can be* η*-simulated with cost* $(n\underline{R}, n\underline{C})$.

III. Multi-Party Quantum Instrument Simulation

In this section, we present two inner bounds (Thms. 1, 2) for the problem of instrument simulation. Owing to the page limit constraints, we exploit the common structural characterization that these three inner bounds possess to provide, compromising readability slightly, a unified characterization. We therefore begin with preliminaries before addressing one-shot simulation with IID codes in Sec. III-A. Defn. 5 below enables us to transition from pre Fourier-Motzkin elimination (FME) bounds to post FME bounds for all three inner bounds.

Defn 5. *Given an ordered collection* $\underline{B} \triangleq (B_1, \cdots, B_6) \in [0,\infty]^6$ *of six non-negative real numbers, we define the region*

$$\alpha(\underline{B}) \triangleq \left\{(R_1, R_2, C_1, C_2) \in \mathbb{R}^4_{\ge 0} : (5) - (8) \text{ below hold}\right\} \tag{4}$$

$$R_1 > B_1 - B_6, R_2 > B_2 - B_6, R_1 + R_2 > B_1 + B_2 - B_6, \tag{5}$$

$$R_1 + C_1 > B_3 - B_6,\ R_1 + R_2 + C_1 > B_2 + B_3 - B_6, \tag{6}$$

$$R_2 + C_2 > B_4 - B_6,\ R_1 + R_2 + C_2 > B_1 + B_4 - B_6, \tag{7}$$

$$R_1 + C_1 + R_2 + C_2 > \max\{B_3 + B_4, B_5\} - B_6. \tag{8}$$

Our next Defn. 6 enables us to characterize all possible "test channels" or single-letter instruments that we can utilize in designing our instrument simulation protocol.

Defn 6. *For instrument* $\mathcal{J} : \mathcal{L}(\mathcal{H}_{\underline{A}}) \longrightarrow \mathcal{L}(\mathcal{H}_{\underline{B}} \otimes \mathbb{C}^{\mathcal{Y}})$ *specified through a collection* $\{\mathcal{J}_y : \mathcal{L}(\mathcal{H}_{\underline{A}}) \to \mathcal{L}(\mathcal{H}_{\underline{B}}) : y \in \mathcal{Y}\}$ *of CP TNI maps, let* $\mathscr{T}(\mathcal{J})$ *denote the collection* $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}})$ *of all triples, where (i)* $\mathcal{I}^{(i)} : \mathcal{L}(\mathcal{H}_{A_i}) \longrightarrow \mathcal{L}(\mathcal{H}_{B_i} \otimes \mathbb{C}^{\mathcal{W}_i})$ *are instruments for* $i \in [2]$ *each specified through collection* $\{\mathcal{I}^{(i)}_{w_i} : \mathcal{L}(\mathcal{H}_{A_i}) \to \mathcal{L}(\mathcal{H}_{B_i}) : w_i \in \mathcal{W}_i\}$ *of CP TNI maps and a stochastic matrix* $p_{Y|W_1W_2}$ *such that* (1) *holds for all* $y \in \mathcal{Y}$*. For a triple* $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}})$*, we let*

$$\sigma^{XB_1B_2W_1W_2Y} \triangleq \left(\mathrm{id}_X \otimes \overline{\mathscr{E}}_p^{Y|\underline{W}} \otimes \underline{\mathcal{I}}\right)\{\varphi^{XA_1A_2}\} \tag{9}$$

$$\sigma_i^{XB_iA_{2-i}W_i} \triangleq \left(\mathrm{id}_{XA_{2-i}} \otimes \mathcal{I}^{(i)}\right)\{\varphi^{XA_1A_2}\} \tag{10}$$

where $\underline{\mathcal{I}} = \left[\mathcal{I}^{(1)} \otimes \mathcal{I}^{(2)}\right]$.

A. Instrument Simulation with IID Codes : One-Shot Regime

Our first finding is an inner bound to distributed instrument simulation in the one-shot regime. Towards characterizing the same, we begin by identifying the one-shot information quantities appearing in the pre FME bounds.

Defn 7. *For* $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \in \mathscr{T}(\mathcal{J})$ *(Defn. 6) and an* $\epsilon > 0$*, define* $\epsilon_5 = 2\log\frac{1}{\epsilon}$, $\underline{\mathsf{B}}^\epsilon \triangleq (\mathsf{B}^\epsilon_1, \mathsf{B}^\epsilon_2, \mathsf{B}^\epsilon_3, \mathsf{B}^\epsilon_4, \mathsf{B}^\epsilon_5, \mathsf{B}^\epsilon_6)$ *where*

$$\mathsf{B}^\epsilon_1 = I^\varepsilon_2(W_1 : A_2X)_{\sigma_1} + \epsilon_5,\ \mathsf{B}^\epsilon_2 = I^\varepsilon_2(W_2 : A_1X)_{\sigma_2} + \epsilon_5$$
$$\mathsf{B}^\epsilon_3 = I^\varepsilon_2(W_1 : Y\underline{B}X)_\sigma + \epsilon_5,\ \mathsf{B}^\epsilon_4 = I^\varepsilon_2(W_2 : Y\underline{B}X)_\sigma + \epsilon_5$$
$$\mathsf{B}^\epsilon_5 = D^\varepsilon_2(\sigma^{XC\underline{BW}Y} \| \sigma^{W_1} \otimes \sigma^{W_2} \otimes \sigma^{XC\underline{B}Y})_\sigma + \epsilon_5,$$
$$\mathsf{B}^\epsilon_6 = I^\varepsilon_H(W_1; W_2)_\sigma - \tfrac{1}{2}\epsilon_5$$

with all the above information quantities computed wrt states in (9)*,* (10)*. Define the region* $\mathscr{A}_\epsilon(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \triangleq \alpha(\underline{\mathsf{B}}^\epsilon)$ *where* $\alpha(\cdot)$ *is defined in Defn. 5.*

We are now set to characterize a one-shot inner bound for multi-party quantum instrument simulation.

Thm 1. *For $\eta > 0$, instrument $\mathcal{J}$'s action on $\rho^{A_1A_2}$ can be η-simulated in one-shot with communication cost (R_1, R_2, C_1, C_2) if there exists a $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \in \mathscr{T}(\mathcal{J})$ for which $(R_1, R_2, C_1, C_2) \in \mathscr{A}_{\tilde{\eta}}(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}})$ where $\tilde{\eta} = \frac{\eta^4}{32^4}$.*

Proof. Fix a generic triple $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \in \mathscr{T}(\mathcal{J})$ and let

$$\mathcal{I}^{(i)}(\rho_{A_i}) = \mathrm{Tr}_{E_i}\left[\sum_{w_i \in \mathcal{W}_i} V_{w_i}\, \rho_{A_i}\, {V_{w_i}}^\dagger \otimes |w_i\rangle\langle w_i|_{W_i}\right],$$

where $V_{w_i} : \mathcal{H}_{A_i} \to \mathcal{H}_{B_i} \otimes \mathcal{H}_{E_i}$, $i \in \{1,2\}$, satisfy $\sum_{w_i \in \mathcal{W}_i} {V_{w_i}}^\dagger V_{w_i} = I_{A_i}$. Let $p_{W_1W_2}(w_1, w_2) \triangleq \mathrm{Tr}\big[(\mu_{w_1} \otimes \mu_{w_2})\, \rho_{A_1A_2}\big], \mu_{w_i} \triangleq {V_{w_i}}^\dagger V_{w_i}$. We begin by describing the K_1, K_2 encoder instruments and the stochastic decoder.

The K_i encoder instruments: Throughout, $K_i = 2^{C_i}, M_i = 2^{R_i}, B_i = 2^{\beta_i} \in \mathbb{N}$ and $[K_i], [M_i], [B_i]$ denote common randomness, message and bin index sets respectively. Since, the two POVM outcomes are correlated, a Slepian-Wolf [14] binning can reduce message rates R_1, R_2. The outcomes are therefore binned and the bin index $b_i \in [B_i]$ is not communicated to the Rx. Specifically, for $i \in [2]$, let $\mathsf{c}^i = (c_{k_i} : k_i \in [K_i])$ denote a multi-code consisting of $K_i = 2^{C_i}$ codes wherein the k_i-th code $c_{k_i} = (w_i(k_i, m_i, b_i) \in \mathcal{W}_i : (m_i, b_i) \in [M_i] \times [B_i])$ consist of $M_i = 2^{R_i}$ bins each with $B_i = 2^{\beta_i}$ codewords. Associated with codeword, $w_i(k_i, m_i, b_i)$, let $V_{k_i,m_i,b_i} \triangleq V_{w_i(k_i,m_i,b_i)}$ $\mu_{k_i,m_i,b_i} \triangleq \mu_{w_i(k_i,m_i,b_i)} \in \mathcal{P}(\mathcal{H}_i)$. Define the operators

$$S_{k_i} \triangleq \sum_{m_i=1}^{M_i} \sum_{b_i=1}^{B_i} \frac{\sqrt{\rho_{A_i}}\, \mu_{k_i,m_i,b_i}\, \sqrt{\rho_{A_i}}}{M_i B_i\, \mathrm{Tr}(\rho_{A_i}\mu_{k_i,m_i,b_i})},\ z_{k_i} \triangleq \Pi_{\mathrm{supp}(S_{k_i})},$$

and $\pi_{k_i} \triangleq I_{A_i} - z_{k_i}$. For $m_i \in [M_i]$ define

$$\theta_{k_i}^{m_i,b_i} \triangleq \frac{V_{k_i,m_i,b_i}\, \sqrt{\rho_{A_i}}\, {S_{k_i}}^{-\frac{1}{2}}}{\sqrt{M_i B_i\, \mathrm{Tr}(\rho_{A_i}\mu_{k_i,m_i,b_i})}}, \tag{11}$$

where ${S_{k_i}}^{-\frac{1}{2}}$ denotes the generalized inverse (equal to 0 on the null space of $S_{k_i}^{(i)}$). Note that, by definition $\theta_{k_i}^{m_i,b_i} : \mathcal{H}_{A_i} \to \mathcal{H}_{B_i} \otimes \mathcal{H}_{E_i}$. Since π_{k_i} is a projector, fix an isometry $Q_i : \mathcal{H}_{A_i} \to \mathcal{H}_{B_i} \otimes \mathcal{H}_{E_i}$ and define the "0-th" operator

$$\theta_{k_i}^0 \triangleq Q_i\, \pi_{k_i}, \qquad \text{so that} \qquad {\theta_{k_i}^0}^\dagger \theta_{k_i}^0 = \pi_{k_i}.$$

Note that for every $(k_i, m_i, b_i) \in [K_i] \times [M_i] \times [B_i]$, we have

$$\theta_{k_i}^0 \sqrt{S_{k_i}} = 0, \theta_{k_i}^{m_i,b_i} \sqrt{S_{k_i}} = \frac{V_{k_i,m_i,b_i} \sqrt{\rho_{A_i}}}{\sqrt{M_i B_i\, \mathrm{Tr}(\rho_{A_i}\mu_{k_i,m_i,b_i})}}.$$

It is also easy to verify that for each $(k_i, m_i, b_i) \in [K_i] \times [M_i] \times [B_i]$, $\theta_{k_i}^{m_i,b_i} : \mathcal{H}_{A_i} \to \mathcal{H}_{B_i} \otimes \mathcal{H}_{E_i}$ and ${\theta_{k_i}^0}^\dagger \theta_{k_i}^0 + \sum_{m_i,b_i} {\theta_{k_i}^{m_i,b_i}}^\dagger \theta_{k_i}^{m_i,b_i} = I_{A_i}$. We are now set to define the two banks of encoding instruments. For $i \in [2]$, Alice$_i$'s instrument corresponding to common randomness index $k_i \in [K_i]$ is $\mathcal{E}_{k_i}^{(i)} : \mathcal{L}(\mathcal{H}_{A_i}) \longrightarrow \mathcal{L}(\mathcal{H}_{B_i} \otimes \mathbb{C}^{M_i})$, defined through its action

$$\mathcal{E}_{k_i}^{(i)}(S) = \mathrm{tr}_{E_i}\left[\begin{array}{r}\displaystyle\sum_{m_i=1}^{M_i} \sum_{b_i=1}^{B_i} \theta_{k_i}^{m_i,b_i}\, S\, {\theta_{k_i}^{m_i,b_i}}^\dagger \otimes |m_i\rangle\langle m_i|_{M_i} \\ +\theta_{k_i}^0\, S\, {\theta_{k_i}^0}^\dagger \otimes |0\rangle\langle 0|_{M_i}\end{array}\right] \tag{12}$$

on any $S \in \mathcal{L}(\mathcal{H}_{A_i})$.

Rem 3. *Observe that $S_{k_i}^{-\frac{1}{2}}$ at the right of $\theta_{k_i}^{m_i,b_i}$ in (11) is a sum spanning over all codebook elements. When we randomize over the codebook, $S_{k_i}^{-\frac{1}{2}}$ will be random. Owing to this random operator sum raised to the power $-\frac{1}{2}$, we are unable to prove any concentration. This causes a central difficulty in analysis which we overcome through the idea of proxy states.*

K_1K_2 Decoder post-processing maps: Recall that the bin indices are not communicated to Charlie. The decoder $\mathcal{D}_{\underline{k}} : \mathcal{L}(\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}) \longrightarrow \mathcal{L}(\mathbb{C}^{|\mathcal{Y}|})$ is therefore a composition $\mathcal{D}_{\underline{k}} \triangleq \mathcal{D}_{\underline{k}}^{(2)} \circ \mathcal{D}_{\underline{k}}^{(1)}$, wherein the $\mathcal{D}_{\underline{k}}^{(1)} : \mathcal{L}(\mathbb{C}^{\underline{M}}) \to \mathcal{L}(\mathbb{C}^{\underline{M}} \otimes \mathbb{C}^{\underline{B}})$ is a standard one-shot hypothesis-testing based [15] decoder used in Slepian Wolf decoding and $\mathcal{D}_{\underline{k}}^{(2)}$ is a plain stochastic processing defined as

$$\mathcal{D}_{\underline{k}}^{(2)}(S) = \sum_{\underline{m} \in [\underline{M}]} \sum_{y \in \mathcal{Y}} p_{Y|\underline{W}}(y|\underline{w}(\underline{k}, \underline{m}, \underline{b}))\, |y\rangle\langle \underline{m}\ \underline{b}|\, S\, |\underline{m}\ \underline{b}\rangle\langle y|\,.$$

for any $S \in \mathcal{L}(\mathbb{C}^{\underline{M}} \otimes \mathbb{C}^{\underline{B}})$ where $\underline{w}(\underline{k}, \underline{m}, \underline{b})$, $\mathbb{C}^{\underline{M}}$, $\mathbb{C}^{\underline{B}}$ abbreviates $w_1(k_1, m_1, b_1), w_2(k_2, m_2, b_2)$, $\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}$, $\mathbb{C}^{B_1} \otimes \mathbb{C}^{B_2}$ respectively. Regarding the former, consider a hypothesis-testing based one-shot Slepian Wolf decoder that tests between $p_{W_1W_2}$ and $p_{W_1} \otimes p_{W_2}$. As this decoder is standard, we omit the details here. A detailed proof is provided in [16] □

B. Instrument Simulation via IID Codes : Asymptotic Regime

Having characterized a one-shot inner bound, it is straightforward to obtain an asymptotic version of the same which we characterize in the following. Rem. 4 highlights the difference in the hypothesis divergence and the one obtained through covering (convex split). We begin by identifying the pre FME information quantities.

Defn 8. *For a triple $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \in \mathscr{T}(\mathcal{J})$ (See Defn. 6) define $\underline{\mathsf{B}}^0 \triangleq (\mathsf{B}_1^0, \mathsf{B}_2^0, \mathsf{B}_3^0, \mathsf{B}_4^0, \mathsf{B}_5^0, \mathsf{B}_6^0)$ where*

$$\begin{array}{l}\mathsf{B}_1^0 = I(W_1 : A_2X)_{\sigma_1},\ \mathsf{B}_2^0 = I(W_2 : A_1X)_{\sigma_2} \\ \mathsf{B}_3^0 = I(W_1 : Y\underline{B}X)_{\sigma},\ \mathsf{B}_4^0 = I(W_2 : Y\underline{B}X)_{\sigma} \\ \mathsf{B}_5^0 = I(W_1, W_2 : Y\underline{B}X)_{\sigma} + I(W_1; W_2)_{\sigma}, \\ \mathsf{B}_6^0 = I(W_1; W_2)_{\sigma}\end{array} \tag{13}$$

with all the above information quantities computed wrt states in (9), (10). Define the region $\mathscr{A}_0(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \triangleq \alpha(\underline{\mathsf{B}}^0)$ where $\alpha(\cdot)$ is defined in Defn. 5.

Thm 2. *Instrument $\mathcal{J}$'s action on $\rho^{A_1A_2}$ can be perfectly simulated with communication cost (R_1, R_2, C_1, C_2) if there exists a $(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}}) \in \mathscr{T}(\mathcal{J})$ for which $(R_1, R_2, C_1, C_2) \in \mathscr{A}_0(\underline{\mathcal{W}}, \underline{\mathcal{I}}, p_{Y|\underline{W}})$*

Rem 4. *In characterizing the inner bound achievable in the asymptotic regime, we note that the maximum appearing in the bound (8) could be simplified. Indeed $\mathsf{B}_5^0 \geq \mathsf{B}_3^0 + \mathsf{B}_4^0$ since $I(X\underline{B}Y; W_1W_2)_\sigma + I(W_1; W_2)_\sigma = I(X\underline{B}Y; W_1)_\sigma + I(X\underline{B}Y; W_2|W_1)_\sigma + I(W_1; W_2)_\sigma = I(X\underline{B}Y; W_1)_\sigma + I(W_1X\underline{B}Y; W_2)_\sigma$. Note that we cannot claim the same inequality in relation to $\mathsf{B}_5^\epsilon, \mathsf{B}_3^\epsilon, \mathsf{B}_4^\epsilon, \mathsf{B}_6^\epsilon$.*

IV. Three Party Purity Distillation

Our main result is an inner bound to the one-shot purity distillation obtained by leveraging Thm 1. Let $\widetilde{H}^{\varepsilon}_{\max}(A|Y)_\rho := \max_{\Omega\subseteq\mathcal{Y}:\, p(\Omega)>1-\sqrt{\varepsilon}} \min_{y\in\Omega} \widetilde{H}^{\varepsilon}_{\max}(A)_{\rho^y_A}$, with $\widetilde{H}^{\varepsilon}_{\max}(A)_{\rho^y_A}$ as in [1].

Thm 3 (Achievable one-shot three-party purity distillation)**.** *Let $\rho_{A_1A_2C} \in \mathcal{D}(\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\mathcal{H}_C)$ and let $0<\varepsilon<1/2$. For a purification φ_ρ of $\rho_{A_1A_2C}$, fix measurement maps (equivalently POVMs) $\mathcal{I}^{(i)} : \mathcal{L}(\mathcal{H}_{A_i}) \to \mathcal{L}(\mathcal{H}_{A_i}\otimes\mathbb{C}^{\mathcal{Y}_i})$, $i\in\{1,2\}$, with finite outcome sets $\mathcal{Y}_1,\mathcal{Y}_2$ and define the cq-states*

$$\omega = \omega_{RA_1A_2CY_1Y_2} := \big(\mathrm{id}_{RC}\otimes\mathcal{I}^{(1)}\otimes\mathcal{I}^{(2)}\big)(\varphi_\rho).$$

$$\omega_i = \omega_{RA_1A_2CY_i} := \big(\mathrm{id}_{RCA_{3-i}}\otimes\mathcal{I}^{(i)}\big)(\varphi_\rho)$$

Fix rates (R_1,R_2,P). If these rates satisfy, for $i\in\{1,2\}$,

$$\begin{aligned} R_i \;&>\; I_2^\varepsilon(Y_i\!:\!RCA_{3-i})_{\omega_i} - I_H^\varepsilon(Y_1;Y_2)_\omega + 3\log\frac{1}{\varepsilon},\\ R_1{+}R_2 \;&>\; I_2^\varepsilon(Y_1\!:\!RCA_2)_{\omega_1} + I_2^\varepsilon(Y_2\!:\!RCA_1)_{\omega_2}\\ &\quad - I_H^\varepsilon(Y_1;Y_2)_\omega + 5\log\frac{1}{\varepsilon},\\ P \le\; &\log d_{A_1}d_{A_2}d_C - \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(A_1|Y_1)_\omega - \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(A_2|Y_2)_\omega\\ &\quad - \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(C|Y_1Y_2)_\omega - I_2^\varepsilon(Y_1\!:\!RA_2C)_{\omega_1}\\ &\quad - I_2^\varepsilon(Y_2\!:\!RA_1C)_{\omega_2} - 4\log\frac{1}{\varepsilon}. \end{aligned} \tag{14}$$

then (R_1,R_2,P) is $14\,\varepsilon^{\frac{1}{8}}$−purity-achievable.

Proof. We only provide a proof sketch here, highlighting the main steps of the argument; a complete and detailed proof can be found in [11]. Assume first that the parties have access to an unlimited amount of common randomness (to be removed later by derandomization). Let $\rho_{A_1A_2C}$ be the shared state, and consider local measurement maps, i.e., instruments

$$\mathcal{I}^{(i)}:\ \mathcal{L}(\mathcal{H}_{A_i}) \longrightarrow \mathcal{L}\big(\mathcal{H}_{A_i}\otimes\mathbb{C}^{\mathcal{Y}_i}\big), \qquad \text{of the form}$$

$$\mathcal{I}^{(i)}(\rho_{A_i}) = \sum_{y_i\in\mathcal{Y}_i} V_{y_i}\rho_{A_i}V_{y_i}^\dagger \otimes |y_i\rangle\langle y_i|_{Y_i}, \qquad \text{where}$$

$V_{y_i}:\mathcal{H}_{A_i}\to\mathcal{H}_{A_i}$, $i\in\{1,2\}$, satisfy $\sum_{y_i\in\mathcal{Y}_i} {V_{y_i}}^\dagger V_{y_i} = I_{A_i}$. In our purity distillation protocol, we require that both the encoders and decoder know the outcome of the protocol; we directly simulate the product instrument $\mathcal{I}^{(1)}\otimes\mathcal{I}^{(2)}$, no stochastic processing is involved at the decoder. Alice$_1$ and Alice$_2$ generate classical messages (m_1,m_2) and Charlie reconstructs outcomes $(\hat{Y}_1,\hat{Y}_2)$ such that the simulated post-instrument state $\widetilde{\omega}$ is η-close (in trace norm) to the ideal post-instrument state $\omega_0 = (\mathrm{id}_C\otimes\mathcal{I}^{(1)}\otimes\mathcal{I}^{(2)})(\rho_{A_1A_2C})$. By monotonicity of the trace distance under local operations, the subsequent local purity distillation steps performed conditioned on $(\hat{Y}_1,\hat{Y}_2)$ incur at most an additional η loss in performance. Thm 1 yields $\eta = 32\epsilon^{\frac{1}{4}}$ with the chosen rate bounds. From defn. $\mathcal{E}^{(i)}_{k_i}$, the proof of Thm 1 shows that an ancilla of size $M_iB_i{+}1$ is required to implement it coherently.

Let $U^{(i)}_{y_i}$ denote the unitary implementing the one-party purity distillation procedure for the conditional state $\rho^{y_i}_{A_i}$ from [1]. For $y=(y_1,y_2)$, define $U_y := U^{(1)}_{y_1}\otimes U^{(2)}_{y_2}$. In [1], the protocol uses an ancillary system whose dimension depends on the amount of purity one aims to distill. Here, we fix a common target purity level that is simultaneously achievable for almost all conditional states $\{\rho^{y_i}_{A_i}\}_{y_i}$, and we implement each unitary $U^{(i)}_{y_i}$ using the corresponding common ancillary register. For those values of y_i for which a larger amount of purity could in principle be distilled, we simply refrain from extracting the additional purity. For the remaining values of y_i for which the target purity is not achievable, we set $U^{(i)}_{y_i}$ to be the identity. Applying unitary $U := \sum_y U_y\otimes|y\rangle\langle y|$ to the state $\widetilde{\omega}$ produced by the instrument simulation protocol is equivalent to the following sequence of operations: coherently implement the encoding instruments $\mathcal{E}^{(i)}_{k_i}$, trace out the systems B_i, apply the controlled unitary $U_k := \sum_m U_{y(k,m,f(k,m))}\otimes|m\rangle\langle m|$, dephase the registers M_1 and M_2, and finally apply the decoding instrument. This decomposition makes explicit that we can (i) coherently realize the protocol instruments, (ii) perform conditional purity distillation on A_1A_2, (iii) transmit the classical registers M_1 and M_2 to Charlie through dephasing channels, and (iv) distill purity from C conditioned on the reconstructed outcomes $(\widetilde{Y}_1,\widetilde{Y}_2)$.

Purity extraction given the outcomes. Now, from the definition of $\widetilde{H}^{\varepsilon}_{\max}(A|Y)_\rho$, a set of measure at least $1-\epsilon^{1/4}$ (with respect to p_{Y_i}) of the states $\rho^{y_i}_{A_i}$ satisfy $\widetilde{H}^{\sqrt{\varepsilon}}_{\max}(A_i|Y_i)_\omega \le \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(A_i)_{\rho^{y_i}_{A_i}}$. Thus using the one-shot single-party purity distillation protocol from [1] on these states, Alice$_i$ distills an amount of local purity on her side of the form $P_{A_i} \le \log d_{A_i} - \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(A_i|Y_i)_\omega$. Since Charlie has access to $(\hat{Y}_1,\hat{Y}_2)$, he can distill local purity from his system at rate $P_C \le \log d_C - \widetilde{H}^{\sqrt{\varepsilon}}_{\max}(C|Y_1Y_2)_\omega$. Each system is $5\epsilon^{1/4}$-close to the target pure state. Summing these contributions and subtracting the purity invested in catalytic pure ancillas used to coherently implement the protocol's instruments yields an overall net distilled purity as in the theorem statement. With the additional common randomness resource, our protocol thus yields a global state $47\epsilon^{\frac{1}{4}}$-close to $|0\rangle\langle0|_{A_1'}\otimes|0\rangle\langle0|_{A_2'}\otimes|0\rangle\langle0|_{C_p}$.

Derandomization. At the end of the previous protocol, the overall action on the system can be summarized as follows:

$$\left\| \frac{1}{K_1K_2}\sum_{k_1,k_2}\mathcal{E}_{k_1k_2}(\rho) - |0\rangle\langle0| \right\|_1 \le \varepsilon_0, \qquad \varepsilon_0 := 47\,\varepsilon^{\frac{1}{4}}.$$

Since $\mathcal{E}_{k_1k_2}$ is CPTP, fidelity-trace distance relation [12, (9.164)] yields

$$\frac{1}{K_1K_2}\sum_{k_1,k_2}\langle0|\,\mathcal{E}_{k_1k_2}(\rho)\,|0\rangle \;\ge\; \Big(1-\frac{\varepsilon_0}{2}\Big)^2.$$

Therefore, there exist indices (k_1^*,k_2^*) such that $\langle0|\,\mathcal{E}_{k_1^*k_2^*}(\rho)\,|0\rangle \ge \big(1-\frac{\varepsilon_0}{2}\big)^2$. Applying once more the same inequality we have

$$\big\|\mathcal{E}_{k_1^*k_2^*}(\rho) - |0\rangle\langle0|\big\|_1 \;\le\; 2\sqrt{\varepsilon_0} = 2\sqrt{47}\,\varepsilon^{\frac{1}{8}} \;\le\; 14\,\varepsilon^{\frac{1}{8}}.$$

Thus, we removed common randomness while keeping the same amount of distilled purity. □

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