A Simple Universal Technique to Analyze Binning over Quantum Channels with Several Applications

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***Abstract*—We propose a simple universal technique based on a likelihood encoder and a soft covering analysis to analyze the canonical strategy of binning that is widely used in channel coding. Through its simplicity and generality, our technique enables us to analyze binning in all known classical and quantum channel coding scenarios. Our technique overcomes the drawbacks of previously developed approaches to analyze binning such as overcounting, non-unique decoding and enables us to derive new inner bounds for 3-user channels.**

I. Introduction

Addressing the problem of channels with state, Gelfand and Pinsker [1] invented a powerful technique of *binning*, wherein a channel code is partitioned into bins. This technique provides an encoder with an exponential collection of codewords - a *bin* - to encode the message. Following its invention, binning has found utility in several fundamental scenarios including the broadcast [2], 3-user channels [3]–[5], channels with state in both classical and quantum [6], [7] realms.

Binning has been proven to yield increased communication rates in any channel coding scenario wherein the channel's operation is affected either by a state [1], [6] or when the encoder must multiplex bit streams [2], [8]. In such scenarios, a bin enables the encoder to intelligently choose the transmitted/chosen codeword to be compatible (typical) with the state [1] or codewords encoding the other receiver (Rx) messages [2]. While this enhances bit rates, its analysis gets involved since the choice implies the transmitted/chosen codeword, and the bin of codewords it is chosen from, are statistically *dependent*. While information-theoretic analysis of binning for classical channels is well studied with several elegant ideas [9], the same does not hold for quantum channels. Specifically, while the event wherein the decoder does not correctly identify the chosen codeword within the bin can be done away with, a quantum analysis forces one to upper bound the probability of this event.

This above mentioned challenge has resulted in multiple attempts and missteps [8, See note in Sec. 1]. In 2015, Savov and Wilde [8] proposed an *overcounting trick* to address this challenge in the context of a $2-$user classical quantum broadcast channel ($2-$CQBC). While the *overcounting trick* resolved the $2-$CQBC case, recent studies [4] assert that the overcounting trick of [8] is inapplicable in more generalized scenarios such as the $3-$CQBC. The following extract from [4] summarizes this. *"First it* (overcounting) *is more involved. Second, the resulting preliminary rate inequalities do not mirror their classical counterparts. And third, it is not clear how it can be extended to the case with a common message (though [40] deals with Marton's code with common message)."* To address the purported difficulties, [4] proposes a non-unique decoding-based approach. Essentially, by pooling together projection operators corresponding to codewords in a bin, [4] claims the challenges with analyzing binning can be overcome. [4] builds on this to present several inner bounds to the $3-$CQBC with multi-level messages analogous to [10]. A careful consideration of the proposed non-unique decoding-based approach reveals concerns in its applicability and the veracity of the proofs.[1] See Rem. 4. In essence, while we are confident the inner bounds stated in [4] hold, the proofs provided in [4] are not complete. This leaves the problem of developing a simple general technique to analyze binning open for classical quantum (CQ) channels.

The modest goal of this article is to put forth and thereby document a simple universal technique to analyze binning that is applicable for any scenario that we are aware of, including classical and CQ channels with state, CQBCs with any number of Rxs, coset coding strategies, etc. Our technique is based on using a likelihood encoder and a soft covering analysis. Our approach separates the encoding technique of binning, i.e. covering with an exponential number of codewords, and decoder packing in such a way that makes it amenable for any scenario. To illustrate this, we present a detailed proof of achievability of Marton's inner bound for the $2-$CQBC in Sec. II. Remark 3 clearly highlights why this approach works for any scenario we are aware of. As an application of our technique, we present, in Sec. III, inner bounds for the $3-$CQBC that combines coset codes, unstructured IID codes, and simultaneous decoding with our likelihood encoder coupled with the soft-covering analysis.

While our techniques are simple, we are not aware of this use of soft covering in a channel coding scenario to analyze binning. We refer to closely related works [11], [12] and [13] that are primarily developed for source coding problems. Here, we remark that for some reason these ideas have not percolated into quantum studies, as evidenced by the missteps as recently as [4]. Simply put, the main utility of our work is possibly a clear documentation of our proposed technique for analyzing binning, thereby preventing missteps.

Notation: For $n \in \mathbb{N}$, $[n] \triangleq \{1, \cdots, n\}$. For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H}), \mathcal{P}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ denote the collection of linear, positive, and density operators acting on $\mathcal{H}$ respectively. We

[1] We have communicated the same to the first author of [4].

let an underline denote an appropriate aggregation of objects. For example, $\underline{\mathcal{V}} \triangleq \mathcal{V}_1 \times \mathcal{V}_2$ for sets, $\mathcal{H}_{\underline{Y}} \triangleq \otimes_{i=1}^{2}\mathcal{H}_{Y_i}$ for Hilbert spaces $\mathcal{H}_{Y_i} : i \in [2]$ etc. We abbreviate probability mass function, random variables, conditional typical projector, and unconditional typical projector as PMF, RVs, C-Typ-Proj and U-Typ-Proj respectively.

II. Binning analysis in the 2−CQBC

Consider a (generic) 2-*CQBC* $(\rho_x \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : x \in \mathcal{X}, \kappa)$ specified through (i) a finite set $\mathcal{X}$, (ii) Hilbert spaces $\mathcal{H}_{Y_j} : j \in [2]$, (iii) a collection $(\rho_x \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : x \in \mathcal{X})$ and (iv) a cost function $\kappa : \mathcal{X} \to [0, \infty)$. The cost function is assumed to be additive, i.e., the cost of preparing the state $\otimes_{t=1}^{n}\rho_{x_t}$ is $\overline{\kappa}^n(x^n) \triangleq \frac{1}{n}\sum_{t=1}^{n}\kappa(x_t)$. Reliable communication on a 2-CQBC entails identifying a code.

Defn 1. *A 2-CQBC code $c = (n, \underline{\mathcal{M}}, e, \underline{\lambda})$ consists of (i) two message index sets $\mathcal{M}_j : j \in [2]$, (ii) an encoder map $e : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ and (iii) POVMs $\lambda_j \triangleq \{\lambda_{j,m_j} \in \mathcal{P}(\mathcal{H}_{Y_j}^{\otimes n}) : m_j \in \mathcal{M}_j\} : j \in [2]$. The average probability of error of the 2-CQBC code $(n, \underline{\mathcal{M}}, e, \underline{\lambda})$ is*

$$\overline{\xi}(e, \underline{\lambda}) \triangleq 1 - \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|}\sum_{\underline{m}\in\underline{\mathcal{M}}}\mathrm{tr}\Big(\lambda_{\underline{m}}\rho_{c,\underline{m}}^{\otimes n}\Big),$$

where $\lambda_{\underline{m}} \triangleq \otimes_{j=1}^{2}\lambda_{j,m_j}$, $\rho_{c,\underline{m}}^{\otimes n} \triangleq \otimes_{t=1}^{n}\rho_{x_t}$ and $(x_t : 1 \le t \le n) = x^n(\underline{m}) \triangleq e(\underline{m})$. Average cost per symbol of transmitting message $\underline{m} \in \underline{\mathcal{M}}$ is $\tau(e|\underline{m}) \triangleq \overline{\kappa}^n(e(\underline{m}))$ and the average cost per symbol of 2-CQBC code is $\tau(e) \triangleq \frac{1}{|\underline{\mathcal{M}}|}\sum_{\underline{m}\in\underline{\mathcal{M}}}\tau(e|\underline{m})$.

Defn 2. *A rate-cost triple $(R_1, R_2, \tau) \in [0, \infty)^3$ is achievable if there exists a sequence of 2-CQBC codes $(n, \underline{\mathcal{M}}^{(n)}, e^{(n)}, \underline{\lambda}^{(n)})$ for which $\lim_{n\to\infty}\overline{\xi}(e^{(n)}, \underline{\lambda}^{(n)}) = 0$,*

$$\lim_{n\to\infty} n^{-1}\log|\mathcal{M}_j^{(n)}| = R_j : j \in [2], \text{ and } \lim_{n\to\infty}\tau(e^{(n)}) \le \tau.$$

The capacity region $\mathscr{C}$ of a 2−CQBC is the set of all achievable rate-cost vectors and $\mathscr{C}(\tau) \triangleq \{\underline{R} : (\underline{R}, \tau) \in \mathscr{C}\}$.

Theorem 1. *A rate-cost triple $(R_1, R_2, \tau) \in [0, \infty)^3$ is achievable if there exists (i) two finite sets $\mathcal{V}_1$, $\mathcal{V}_2$, (ii) a PMF $p_{V_1V_2}$ on $\mathcal{V}_1\times\mathcal{V}_2$ and (iii) a mapping function $f : \mathcal{V}_1\times\mathcal{V}_2 \to \mathcal{X}$ such that $\sum_{(v_1,v_2)} p_{V_1V_2}(v_1, v_2)\kappa(f(v_1, v_2)) \le \tau$,*

$$R_1 + R_2 < \sum_{j=1}^{2} I(V_j; Y_j) - I(V_1; V_2) \text{ and } R_j < I(V_j; Y_j)$$

holds, for $j \in [2]$, where all the information quantities are computed wrt the state $\rho^{\underline{V}X\underline{Y}} \triangleq \sum_{(\underline{v},x)} p_{\underline{V}X}(\underline{v}, x)\,|\underline{v}, x\rangle\langle\underline{v}, x| \otimes \rho_x^{Y_1Y_2}$, with $p_{\underline{V}X}(\underline{v}, x) = p_{\underline{V}}(\underline{v})\mathbb{1}\{x = f(\underline{v})\}$, for $(\underline{v}, x) \in \underline{\mathcal{V}} \times \mathcal{X}$.

Remark 1. *Note that we present our approach only for binning. Our technique and its analysis straightforwardly extend to the general coding scheme involving both binning and superposition.*

Proof. Consider a pair $\mathcal{V}_j : j \in [2]$ of finite sets, a generic function $f : \underline{\mathcal{V}} \to \mathcal{X}$ and a generic PMF $p_{\underline{V}}$ on $\underline{\mathcal{V}}$ that satisfy the average cost $\sum_{\underline{v}} p_{\underline{V}}(\underline{v})\kappa(f(\underline{v})) \le \tau$. Let (R_1, R_2) be a rate pair that satisfy the corresponding bounds in Thm. 1. We divide the proof into four parts entailing the code structure, encoding rule, decoding POVMs and error analysis.

Code structure: For $j \in [2]$, the code employed to communicate Rx j's message comprises (i) a collection $\mathcal{C}_j \triangleq \{v_j^n(b_j, m_j) : b_j \in [2^{nK_j}], m_j \in [2^{nR_j}]\}$ constructed over the alphabet $\mathcal{V}_j^n$ and (ii) a binning map $b_j : [2^{nR_1}] \times [2^{nR_2}] \to [2^{nK_j}]$. For $j \in [2]$, let $c_j(m_j) \triangleq (v_j^n(b_j, m_j) : b_j \in [2^{nK_j}])$ denote the bin corresponding to message m_j. Conventionally, the codewords in $\mathcal{C}_j$ are picked IID with distribution $\prod_{t=1}^{n} p_{V_j}$. However, to demonstrate the generality of our approach, we pick these codewords, as will be formalized when we analyze the error probability, IID with an alternate distribution $\prod_{t=1}^{n} q_{V_j}$. Secondly, the binning map, which identifies the chosen codewords, is conventionally based on typicality. In other words, $(b_1(\underline{m}), b_2(\underline{m})) \in [2^{nK_1}] \times [2^{nK_2}]$ is chosen deterministically to ensure $v_j^n(b_j(\underline{m}), m_j) : j \in [2]$ is jointly typical with respect to $p_{V_1V_2}$. A corresponding randomization - uniformly chosen amongst all joint typical pairs - is employed to analyze error probability. Instead of forcing typicality, as is done conventionally, we employ a soft probabilistic approach inspired by Cuff's likelihood encoder [14] to determine our binning. This will be evident when we specify the distribution with respect to which the binning map $b_j : [2^{nR_1}] \times [2^{nR_2}] \to [2^{nK_j}]$ is chosen.

Encoding Rule: For each message pair (m_1, m_2), the encoder chooses codeword pair $\big(v_j^n(b_j(\underline{m}), m_j) : j \in [2]\big)$. To communicate the messages (m_1, m_2), the encoder prepares the quantum state $\rho_{x^n(\underline{m})}^{Y_1Y_2}$ where $x^n(\underline{m}) \triangleq f^n\left(v_1^n(b_1(\underline{m}), m_1), v_2^n(b_2(\underline{m}), m_2)\right)$, with f evaluated letter-by-letter.

Decoding POVMs: Since the decoding POVMS for Rx 1 and Rx 2 being identical, we present the same in terms of a generic index j. Let $\pi_{v_j^n}$ and π denote the C-TYP-Proj and U-TYP-Proj with respect to the state $\rho_{v_j^n}$ and ρ respectively, where $\rho_{v_j^n} = \bigotimes_{t=1}^{n}\rho_{v_{jt}}$ and for all $v_j \in \mathcal{V}_j$, $\rho_{v_j} \triangleq \sum_{v_{\bar{j}}} p_{V_{\bar{j}}|V_j}(v_{\bar{j}}|v_j)\rho_{f(v_1,v_2)}^{Y_j}$, where $\bar{j} \in \{1,2\}\backslash\{j\}$ denotes the complement index, i.e., $\{j, \bar{j}\} = \{1, 2\}$, and $\rho \triangleq \sum_{v_1,v_2} p_{V_1V_2}(v_1, v_2)\rho_{f(v_1,v_2)}^{Y_j}$. We define the square-root measurement $\{\lambda_{m_j} : m_j \in [2^{nR_j}]\}$ to decode m_j as

$$\lambda_{m_j} \triangleq \left(\sum_{(\tilde{b}_j,\tilde{m}_j)}\gamma_{\tilde{b}_j,\tilde{m}_j}\right)^{-\frac{1}{2}}\left(\sum_{b_j}\gamma_{b_j,m_j}\right)\left(\sum_{(\tilde{b}_j,\tilde{m}_j)}\gamma_{\tilde{b}_j,\tilde{m}_j}\right)^{-\frac{1}{2}},$$

and $\lambda_{-1}^{Y_j} \triangleq I - \sum_{m_j}\lambda_{m_j}$, where $\gamma_{b_j,m_j} \triangleq \pi\pi_{v_j^n(b_j,m_j)}\pi$.

Error analysis: We begin by specifying the codebook distribution. Let

$$P\begin{pmatrix}\left(V_j^n(b_j, m_j) = v_j^n(b_j, m_j) : b_j \in [2^{nK_j}], m_j \in [2^{nR_j}]\right), \\ \left(B_j(\underline{m}) = b_j(\underline{m}) : \underline{m} \in [2^{nR_1}] \times [2^{nR_2}]\right) : j \in [2]\end{pmatrix}$$
$$= \left[\prod_{m_1=1}^{2^{nR_1}}\prod_{b_1=1}^{2^{nK_1}} q_{V_1}^n(v_1^n(b_1, m_1))\right]\left[\prod_{m_2=1}^{2^{nR_2}}\prod_{b_2=1}^{2^{nK_2}} q_{V_2}^n(v_2^n(b_2, m_2))\right]$$

$$\left[\prod_{m_1=1}^{2^{nR_1}}\prod_{m_2=1}^{2^{nR_2}}\frac{r^n_{\underline{V}}(v^n_1(b_1(\underline{m}),m_1),v^n_2(b_2(\underline{m}),m_2))}{\sum_{(l_1,l_2)}r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}\right], \quad (1)$$

specify our joint codebook distribution, where

$$r^n_{\underline{V}}(v^n_1,v^n_2)\triangleq\frac{p^n_{\underline{V}}(v^n_1,v^n_2)}{q^n_{V_1}(v^n_1)q^n_{V_2}(v^n_2)},\ \text{for } (v^n_1,v^n_2)\in\mathcal{V}_1\times\mathcal{V}_2. \quad (2)$$

Remark 2. *As specified when we described the code structure, the distribution of our binning map conditioned on the code distinguishes our encoder from conventional joint typicality-based encoders. Specifically, note from* (1) *that* $P\left(B_j(\underline{m})=b_j:j\in[2]\middle|C_j(m_j)=c_j(m_j):j\in[2]\right)=$

$$\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{\sum_{(l_1,l_2)}r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}.$$

We now derive an upper bound on the average error probability. We have

$$\bar{\xi}(e,\underline{\lambda})\triangleq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\operatorname{tr}\left\{(I^{Y_1Y_2}-\lambda_{m_1}\otimes\lambda_{m_2})\rho^{Y_1Y_2}_{e(\underline{m})}\right\}$$

$$\leq T_1+T_2,\ \text{where, for } j\in[2]$$

$$T_j\triangleq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\operatorname{tr}\left\{\left(I^{Y_j}-\Gamma_{m_j}\right)\rho^{Y_j}_{e(\underline{m})}\right\},\text{ and}$$

$$\Gamma_{m_j}\triangleq\left(\sum_{(\tilde{b}_j,\tilde{m}_j)}\gamma_{\tilde{b}_j,\tilde{m}_j}\right)^{-\frac{1}{2}}\gamma_{b_j(\underline{m}),m_j}\left(\sum_{(\tilde{b}_j,\tilde{m}_j)}\gamma_{\tilde{b}_j,\tilde{m}_j}\right)^{-\frac{1}{2}}.$$

The inequality follows from (i) $I^{Y_1Y_2}-\lambda_{m_1}\otimes\lambda_{m_2}\leq(I^{Y_1Y_2}-\lambda_{m_1}\otimes I^{Y_2})+(I^{Y_1Y_2}-I^{Y_1}\otimes\lambda_{m_2})$ and (ii) the fact that $\lambda_{m_j}\geq\Gamma_{m_j}$, for $j\in[2]$. Observe that,

$$T_j=\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\mathbb{1}\{b_1(\underline{m})=b_1,b_2(\underline{m})=b_2\}$$
$$\operatorname{tr}\left\{\left(I^{Y_j}-\Gamma_{m_j}\right)\rho^{Y_j}_{f^n\left(v^n_1(b_1,m_1),v^n_2(b_2,m_2)\right)}\right\}$$
$$\stackrel{(*)}{=}T_{j.1}+T_{j.2},\ \text{where}$$
$$T_{j.1}\triangleq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}$$
$$\left[\mathbb{1}\{b_1(\underline{m})=b_1,b_2(\underline{m})=b_2\}-\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}\right]$$
$$\operatorname{tr}\left\{\left(I^{Y_j}-\Gamma_{m_j}\right)\rho^{Y_j}_{f^n\left(v^n_1(b_1,m_1),v^n_2(b_2,m_2)\right)}\right\}$$
$$\text{and } T_{j.2}\triangleq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}$$
$$\operatorname{tr}\left\{\left(I^{Y_j}-\Gamma_{m_j}\right)\rho^{Y_j}_{f^n\left(v^n_1(b_1,m_1),v^n_2(b_2,m_2)\right)}\right\}.$$

The equality $(*)$ follows by adding and subtracting

$$\frac{r^n_{\underline{V}}\left(v^n_1(b_1,m_1),v^n_2(b_2,m_2)\right)}{2^{n(K_1+K_2)}}, \quad (3)$$

thereby decomposing T_j into $T_{j.1}$ and $T_{j.2}$ described above.

Remark 3. *We highlight the above technique of adding and subtracting the term in* (3) *which enables us to decompose the error into two terms, $T_{j.1}$ is analyzed using the soft-covering lemma and $T_{j.2}$ using a conventional channel coding approach. This enables us to break the statistical dependence. Moreover, bounding $\frac{q_{V_j}(.)}{p_{V_j}(.)}$ appropriately via the divergence (see equation* (4)*) enable us to obtain the right channel coding bound. This approach essentially decouples the binning analysis and the channel coding analysis, thus leaving the channel coding analysis oblivious to the presence of binning.*

The term $T_{j.1}$ is simply bounded by a covering argument [15] referred to therein as 'cloud mixing'. Observe that,

$$T_{j.1}\leq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}$$
$$\left|\mathbb{1}\{b_1(\underline{m})=b_1,b_2(\underline{m})=b_2\}-\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}\right|$$
$$\left|\operatorname{tr}\left\{\left(I^{Y_j}-\Gamma_{m_j}\right)\rho^{Y_j}_{f^n\left(v^n_1(b_1,m_1),v^n_2(b_2,m_2)\right)}\right\}\right|$$
$$\leq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}$$
$$\left|\mathbb{1}\{b_1(\underline{m})=b_1,b_2(\underline{m})=b_2\}-\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}\right|.$$

We evaluate the expectation over $C_j(m_j):j\in[2]$. We obtain

$$\mathbb{E}[T_{j.1}]\leq\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{c_j(m_j):j\in[2]}$$
$$P\left(C_j(m_j)=c_j(m_j):j\in[2]\right)$$
$$\left|\begin{array}{l}P\left(B_j(\underline{m})=b_j:j\in[2]\middle|C_j(m_j)=c_j(m_j):j\in[2]\right)\\-\dfrac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}\end{array}\right|$$
$$=\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{c_j(m_j):j\in[2]}P\left(C_j(m_j)=c_j(m_j):j\in[2]\right)$$
$$\left|\begin{array}{r}\dfrac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{\sum_{(l_1,l_2)}r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}\\-\dfrac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{2^{n(K_1+K_2)}}\end{array}\right|$$
$$=\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{c_j(m_j):j\in[2]}P\left(C_j(m_j)=c_j(m_j):j\in[2]\right)$$
$$\frac{r^n_{\underline{V}}(v^n_1(b_1,m_1),v^n_2(b_2,m_2))}{\sum_{(l_1,l_2)}r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}$$
$$\left|1-\sum_{(l_1,l_2)}\frac{r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}{2^{n(K_1+K_2)}}\right|$$
$$=\frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{c_j(m_j):j\in[2]}P\left(C_j(m_j)=c_j(m_j):j\in[2]\right)$$
$$\left|\sum_{(l_1,l_2)}\frac{r^n_{\underline{V}}(v^n_1(l_1,m_1),v^n_2(l_2,m_2))}{2^{n(K_1+K_2)}}-1\right|$$

$$= E\left\{\left|\sum_{(l_1,l_2)} \frac{r^n_{\underline{V}}(V_1^n(l_1,M_1),V_2^n(l_2,M_2))}{2^{n(K_1+K_2)}} - 1\right|\right\},$$

From the 'cloud mixing' lemma [15, Lemma 19], we have $\mathbb{E}[T_{j.1}] \leq \epsilon$, if $K_1 + K_2 > D(p_{V_1V_2}\|q_{V_1}q_{V_2})$ and $K_j > D(p_{V_j}\|q_{V_j}),$ for $j \in [2]$.

Remark 4. *Next, we employ the Hayashi Nagaoka inequality [16]. We now elaborate on the mis-steps in [4]. The operator $\Pi^{B1}_{u_1(m_0,m_{11},m_{21},m_{12})}$ defined in [4, Eq.4] is identified as S in the Hayashi Nagaoka inequality. Note, however, that the hypothesis requires $0 \leq S \leq I$. Since $\Pi^{B1}_{u_1(m_0,m_{11},m_{21},m_{12})}$ is a <u>sum</u> of projectors, $S \leq I$ is <u>not</u>. Indeed, if the codewords in the bin are identical, which is not precluded via a random code construction, $\Pi^{B1}_{u_1(m_0,m_{11},m_{21},m_{12})}$ will not be dominated by I.*

Continuing with our proof, we bound the term $T_{j.2}$ on the above using the Hayashi Nagaoka inequality [16]. Specifically, we identify $\gamma_{b_j(\underline{m}),m_j} \triangleq \pi\pi_{v_j^n(b_j(\underline{m}),m_j)}\pi$ as S. This identification is justified, since $\gamma_{b_j(\underline{m}),m_j} = \pi\pi_{v_j^n(b_j(\underline{m}),m_j)}\pi \leq \pi \leq I$. We obtain $T_{j.2} \leq 2\,T_{j.2.1} + 4\,(T_{j.2.2} + T_{j.2.3}),$ where

$$T_{j.2.1} \triangleq \frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}} \frac{r^n_{\underline{V}}(v_1^n(b_1,m_1),v_2^n(b_2,m_2))}{2^{n(K_1+K_2)}}$$
$$\operatorname{tr}\left\{(I^{Y_j} - \pi\,\pi_{v_j^n(b_j,m_j)}\,\pi)\rho^{Y_j}_{f^n\left(v_1^n(b_1,m_1),v_2^n(b_2,m_2)\right)}\right\},$$
$$T_{j.2.2} \triangleq \frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{\tilde{m}_j\neq m_j}\sum_{\tilde{b}_j}$$
$$\frac{r^n_{\underline{V}}(v_1^n(b_1,m_1),v_2^n(b_2,m_2))}{2^{n(K_1+K_2)}}$$
$$\operatorname{tr}\left\{\pi\,\pi_{v_j^n(\tilde{b}_j,\tilde{m}_j)}\,\pi\rho^{Y_j}_{f^n\left(v_1^n(b_1,m_1),v_2^n(b_2,m_2)\right)}\right\}, \text{ and}$$
$$T_{j.2.3} \triangleq \frac{1}{2^{n(R_1+R_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{\tilde{b}_j\neq b_j} \frac{r^n_{\underline{V}}(v_1^n(b_1,m_1),v_2^n(b_2,m_2))}{2^{n(K_1+K_2)}}$$
$$\operatorname{tr}\left\{\pi\,\pi_{v_j^n(\tilde{b}_j,m_j)}\,\pi\rho^{Y_j}_{f^n\left(v_1^n(b_1,m_1),v_2^n(b_2,m_2)\right)}\right\}.$$

<u>*Analysis of $T_{j.2.1}$:*</u> We evaluate the expectation of $T_{j.2.1}$ over the codebook generation distribution. We obtain

$$\mathbb{E}[T_{j.2.1}] = \frac{1}{2^{n(R_1+R_2)}}\frac{1}{2^{n(K_1+K_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{v_1^n,v_2^n} q^n_{V_1}(v_1^n)$$
$$q^n_{V_2}(v_2^n)r^n_{\underline{V}}(v_1^n,v_2^n)\operatorname{tr}\left\{\left(I^{Y_j} - \pi\,\pi_{v_j^n}\,\pi\right)\rho^{Y_j}_{f^n(v_1^n,v_2^n)}\right\}$$
$$\overset{(a)}{=} \frac{1}{2^{n(R_1+R_2)}}\frac{1}{2^{n(K_1+K_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{v_1^n}\sum_{v_2^n} p^n_{\underline{V}}(v_1^n,v_2^n)$$
$$\operatorname{tr}\left\{\left(I^{Y_j} - \pi\,\pi_{v_j^n}\,\pi\right)\rho^{Y_j}_{f^n(v_1^n,v_2^n)}\right\}$$
$$\overset{(b)}{=} \frac{1}{2^{n(R_j+K_j)}}\sum_{m_j}\sum_{b_j}\sum_{v_j^n} p^n_{V_j}(v_j^n)\operatorname{tr}\left\{\left(I^{Y_j} - \pi\,\pi_{v_j^n}\,\pi\right)\rho^{Y_j}_{v_j^n}\right\}$$
$$= \frac{1}{2^{n(R_j+K_j)}}\sum_{m_j}\sum_{b_j}\sum_{v_j^n\notin T^n_\delta(p_{V_j})} p^n_{V_j}(v_j^n) + \frac{1}{2^{n(R_j+K_j)}}\sum_{m_j}$$
$$\sum_{b_j}\sum_{v_j^n\in T^n_\delta(p_{V_j})} p^n_{V_j}(v_j^n)\operatorname{tr}\left\{\left(I^{Y_j} - \pi\,\pi_{v_j^n}\,\pi\right)\rho^{Y_j}_{v_j^n}\right\}$$
$$\leq \epsilon + \frac{1}{2^{n(R_j+K_j)}}\sum_{m_j}\sum_{b_j}\sum_{v_j^n\in T^n_\delta(p_{V_j})} p^n_{V_j}(v_j^n)$$
$$\operatorname{tr}\left\{\left(I^{Y_j} - \pi\,\pi_{v_j^n}\,\pi\right)\rho^{Y_j}_{v_j^n}\right\} \overset{(c)}{\leq} 2\epsilon + 2\sqrt{\epsilon},$$

where (a) follows from (2), (b) follows from the definition of $\rho^{Y_j}_{v_j^n}$, and (c) follows from $\operatorname{tr}\left\{\pi\pi_{v_j^n}\pi\rho^{Y_j}_{v_j^n}\right\} = \operatorname{tr}\left\{\pi_{v_j^n}\pi\rho^{Y_j}_{v_j^n}\pi\right\} \geq \operatorname{tr}\left\{\pi_{v_j^n}\rho^{Y_j}_{v_j^n}\right\} - \left\|\pi\rho^{Y_j}_{v_j^n}\pi - \rho^{Y_j}_{v_j^n}\right\|_1 \geq 1 - \epsilon - 2\sqrt{\epsilon}$, where the first equality follows from the cyclicity of the trace, first inequality follows from using $\operatorname{tr}(\Delta\rho) \geq \operatorname{tr}(\lambda\sigma) - \|\rho-\sigma\|_1$, for $0 \leq \Delta,\rho,\sigma \leq I$, and second inequality follows from the typical property $\operatorname{tr}\left\{\pi_{v_j^n}\rho^{Y_j}_{v_j^n}\right\} \geq 1-\epsilon$, for $v_j^n \in T^n_\delta(p_{V_j})$ and from the gentle measurement operator [17].

<u>*Analysis of $T_{j.2.2}$:*</u> We evaluate the expectation of $T_{j.2.2}$ over the codebook generation distribution. We obtain

$$\mathbb{E}[T_{j.2.2}] = \frac{1}{2^{n(R_1+R_2)}}\frac{1}{2^{n(K_1+K_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{\tilde{m}_j\neq m_j}\sum_{\tilde{b}_j}\sum_{\tilde{v}_j^n}\sum_{\underline{v}^n}$$
$$q^n_{V_j}(\tilde{v}_j^n)\,q^n_{V_1}(v_1^n)\,q^n_{V_2}(v_2^n)\,r^n_{\underline{V}}(v_1^n,v_2^n)\operatorname{tr}\left\{\pi\pi_{\tilde{v}_j^n}\pi\rho^{Y_j}_{f^n(v_1^n,v_2^n)}\right\}$$
$$\overset{(a)}{=} \frac{1}{2^{n(R_1+R_2)}}\frac{1}{2^{n(K_1+K_2)}}\sum_{\underline{m}}\sum_{\underline{b}}\sum_{\tilde{m}_j\neq m_j}\sum_{\tilde{b}_j}\sum_{\tilde{v}_j^n}\sum_{v_1^n,v_2^n}$$
$$q^n_{V_j}(\tilde{v}_j^n)\,p^n_{\underline{V}}(v_1^n,v_2^n)\operatorname{tr}\left\{\pi\pi_{\tilde{v}_j^n}\pi\rho^{Y_j}_{f^n(v_1^n,v_2^n)}\right\}$$
$$\overset{(b)}{=} \sum_{\tilde{m}_j\neq m_j}\sum_{\tilde{b}_j}\sum_{\tilde{v}_j^n\in T^n_\delta(p_{V_j})} q^n_{V_j}(\tilde{v}_j^n)\operatorname{tr}\left\{\pi\pi_{\tilde{v}_j^n}\pi\left(\rho^{Y_j}\right)^{\otimes n}\right\}$$
$$\overset{(c)}{\leq} \sum_{\tilde{m}_j}\sum_{\tilde{b}_j}\sum_{\tilde{v}_j^n\in T^n_\delta(p_{V_j})} 2^{-n(D(p_{V_j}||q_{V_j})-\delta_j)}\,p^n_{V_j}(\tilde{v}_j^n)$$
$$\operatorname{tr}\left\{\pi_{\tilde{v}_j^n}\,\pi\left(\rho^{Y_j}\right)^{\otimes n}\pi\right\} \tag{4}$$
$$\overset{(d)}{\leq} 2^{-n\left(I(V_j;Y_j)+D(p_{V_j}||q_{V_j})-2\delta-\delta_j-(R_j+K_j)\right)},$$

where (a) follows from (2), (b) follows from the definition of $\left(\rho^{Y_j}\right)^{\otimes n}$, and from $\pi_{\tilde{v}_j^n} = 0$, for $\tilde{v}_j^n \notin T^n_\delta(p_{V_j})$, (c) follows from (i) the cyclicity of the trace and from (ii) $q^n_{V_j}(\tilde{v}_j^n) \leq 2^{-n\left(D(p_{V_j}||q_{V_j})-\delta_j\right)}\,p^n_{V_j}(\tilde{v}_j^n)$, for $\tilde{v}_j^n \in \mathcal{T}^n_\delta(p_{V_j})$, (d) follows from $\operatorname{tr}\left\{\pi_{\tilde{v}_j^n}\pi\left(\rho^{Y_j}\right)^{\otimes n}\pi\right\} \leq 2^{-n(H(Y_j)-\delta)}\operatorname{tr}\left\{\pi_{\tilde{v}_j^n}\pi\right\} \leq 2^{-n(H(Y_j)-\delta)}\operatorname{tr}\left\{\pi_{\tilde{v}_j^n}\right\} \leq 2^{-n(H(Y_j)-\delta)}2^{n(H(Y_j|V_j)-\delta)}$, for $\tilde{v}_j^n \in T^n_\delta(p_{V_j})$. Hence, for $\epsilon > 0$, $\mathbb{E}[T_{j.2.2}] \leq \epsilon$, if $R_j + K_j < I(V_j;Y_j) + D(p_{V_j}||q_{V_j})$. Following the identical steps as used in bounding $\mathbb{E}[T_{j.2.2}]$, for $\epsilon > 0$, we have, $\mathbb{E}[T_{j.2.3}] \leq \epsilon$, if $K_j < I(V_j;Y_j)+D(p_{V_j}||q_{V_j})$. This completes the proof. The bounds stated in Thm. 1 can now be derived via Fourier–Motzkin elimination [9, Appendix D]. ☐

III. New Achievable Rate Regions for 3−CQBC

Notation: For prime $\upsilon \in \mathbb{Z}$, $\mathcal{F}_\upsilon$ will denote finite field of size $\upsilon \in \mathbb{Z}$ with $\oplus$ denoting field addition in $\mathcal{F}_\upsilon$ (i.e. mod−υ). In any context, when used in conjunction i, j, k will denote

$$\Psi^{\underline{U}\,\underline{U}^{\oplus}\,\underline{V}X\underline{Y}} \triangleq \sum_{\substack{u_{12},u_{13}\\u_{21},u_{23}\\u_{31},u_{32}}} \sum_{\substack{v_1,v_2,v_3,x\\u_1^{\oplus},u_2^{\oplus},u_3^{\oplus}}} p_{\underline{U}\,\underline{V}X}(u_{1*},u_{2*},u_{3*},\underline{v},x)\mathbb{1}_{\left\{\substack{u_{ij}\oplus u_{kj}\\=u_j^{\oplus}:j\in[3]}\right\}} \big|u_{1*}\,u_{2*}\,u_{3*}\,u_1^{\oplus}\,u_2^{\oplus}\,u_3^{\oplus}\,\underline{v}\,x\big\rangle\big\langle u_{1*}\,u_{2*}\,u_{3*}\,u_1^{\oplus}\,u_2^{\oplus}\,u_3^{\oplus}\,\underline{v}\,x\big|\otimes\rho_x$$

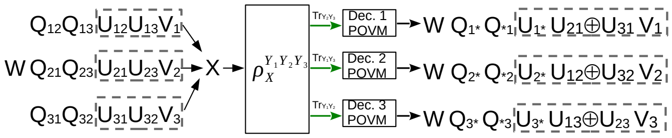

Fig. 1. Depiction of all RVs in the full blown coding strategy. In Sec. IV-A (Step I) only RVs in the grey dashed box are non-trivial, with the rest trivial.

distinct indices in $[3]$, hence $\{i,j,k\}=[3]$ and we let $[\![3]\!] \triangleq \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$. We let an underline denote an appropriate aggregation of objects. For example, $\underline{\mathcal{V}} \triangleq \mathcal{V}_1\times\mathcal{V}_2\times\mathcal{V}_3$ for sets, $\mathcal{H}_{\underline{Y}} \triangleq \otimes_{i=1}^{3}\mathcal{H}_{Y_i}$ for Hilbert spaces $\mathcal{H}_{Y_i}: i\in[3]$ etc.

Consider a (generic) 3-*CQBC* $(\rho_x\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):x\in\mathcal{X},\kappa)$ specified through (i) a finite set $\mathcal{X}$, (ii) Hilbert spaces $\mathcal{H}_{Y_j}: j\in[3]$, (iii) a collection $(\rho_x\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):x\in\mathcal{X})$ and (iv) a cost function $\kappa:\mathcal{X}\to[0,\infty)$. The cost function is assumed to be additive, i.e., the cost of preparing the state $\otimes_{t=1}^{n}\rho_{x_t}$ is $\overline{\kappa}^n(x^n)\triangleq\frac{1}{n}\sum_{t=1}^{n}\kappa(x_t)$. Reliable communication on a 3-CQBC entails identifying a code. We refer to [18] for standard definitions of code, achievable rate region and capacity region.

IV. Simultaneous Decoding of Coset Codes

We employ the following general coding strategy. See Fig. 1. Each Rx j's message is split into six parts $m_j=(m_j^W,m_{ji}^Q,m_{jk}^Q,m_{ji}^U,m_{jk}^U,m_j^V)$. m_j^W,m_{ji}^Q,m_{jk}^Q and m_j^V are encoded using unstructured IID codes built on $\mathcal{W},\mathcal{Q}_{ji},\mathcal{Q}_{jk},\mathcal{V}_j$ respectively. m_{ji}^U and m_{jk}^U are encoded via coset codes built on $\mathcal{U}_{ji}=\mathcal{F}_{\upsilon_i},\mathcal{U}_{jk}=\mathcal{F}_{\upsilon_k}$ respectively. Rx j decodes $\underline{m}^W,m_{ji}^Q,m_{jk}^Q,m_{ji}^U,m_{jk}^U,m_j^V,m_{ij}^Q,m_{kj}^Q$ and the sum $U_{ij}^n(m_{ij}^U)\oplus U_{kj}^n(m_{kj}^U)$ of the codewords corresponding to m_{ij}^U and m_{kj}^U respectively, where $\underline{m}^W\triangleq(m_1^W,m_2^W,m_3^W)$. We let (i) $Q_{j*}\triangleq(Q_{ji},Q_{jk})$, (ii) $Q_{*j}\triangleq(Q_{ij},Q_{kj})$, (iii) $U_{j*}\triangleq(U_{ji},U_{jk})$, and (iv) $U_j^{\oplus}\triangleq U_{ij}\oplus U_{kj}$ denote RVs encoding (i) semi-private parts m_{ji}^Q,m_{jk}^Q of Rx j, (ii) semi-private parts of Rx i,k decoded by Rx j, (iii) bivariate parts m_{ji}^U,m_{jk}^U of Rx j, and (iv) bivariate component decoded by Rx j respectively. We present the analysis of general strategy in two steps. In this article, we focus on the first step involving only coset codes (Thm. 2). See [18] for a complete proof and a characterization of Step II inner bound.

A. Step I : Simultaneous decoding of Bivariate Components

In this first step, we activate only the 'bivariate' $\underline{U}\triangleq(U_{1*},U_{2*},U_{3*})$ and private parts $\underline{V}\triangleq(V_1,V_2,V_3)$ choosing them to be non-trivial, with the rest $\underline{W}=\phi$, $\underline{Q}=(Q_{1*},Q_{2*},Q_{3*})=\phi$ trivial. Since $\underline{V}$ is coded via unstructured IID and $\underline{U}$ is coded via coset codes, Thm. 2 addresses all non-trivial elements.

Theorem 2. *Let $\hat{\alpha}_S\in[0,\infty)^4$ be the set of all rate-cost quadruples $(\underline{R},\tau)\in[0,\infty)^4$ for which there exists (i) finite sets $\mathcal{V}_j:j\in[3]$, (ii) finite fields $\mathcal{U}_{ij}=\mathcal{F}_{\upsilon_j}$ for each $ij\in[\![3]\!]$, (iii) a PMF $p_{\underline{U}\,\underline{V}X}=p_{U_{1*}U_{2*}U_{3*}V_1V_2V_3X}$ on $\underline{\mathcal{U}}\times\underline{\mathcal{V}}\times\mathcal{X}$, (iv) nonnegative numbers $S_{ij},T_{ij}:ij\in[\![3]\!],K_j,L_j:j\in[3]$, such that $R_1=T_{12}+T_{13}+L_1,R_2=T_{21}+T_{23}+L_2,R_3=T_{31}+T_{32}+L_3$,*

$$\mathbb{E}\{\kappa(X)\}\leq\tau, S_A+M_B+K_C \overset{\boldsymbol{\alpha}}{>} \Theta(A,B,C),\ \textit{where} \quad (5)$$

$$\Theta(A,B,C)\triangleq\max_{\substack{(\theta_j:j\in B)\\ \in\prod_{j\in B}\mathcal{F}_{\upsilon_j}}}\left\{\begin{array}{c}\sum_{a\in A}\log|\mathcal{U}_a|+\sum_{j\in B}\log\upsilon_j\\+\sum_{c\in C}H(V_c)\\-H(U_A,U_{ij}\oplus\theta_jU_{kj}:j\in B,V_C)\end{array}\right\}$$

for all $A\subseteq\{12,13,21,23,31,32\},B\subseteq\{1,2,3\},C\subseteq\{1,2,3\}$, that satisfy $A\cap A(B)=\phi$, where $A(B)=\cup_{j\in B}\{ji,jk\}$, $U_A=(U_{jk}:jk\in A)$, $V_C=(V_c:c\in C)$, $S_A=\sum_{jk\in A}S_{jk},M_B\triangleq\sum_{j\in B}\max\{S_{ij}+T_{ij},S_{kj}+T_{kj}\},K_C=\sum_{c\in C}K_c$, and

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|-H(U_{\mathcal{A}_j}|U_{\mathcal{A}_j^c},U_{ij}\oplus U_{kj},V_j,Y_j)$$

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}+S_{ij}+T_{ij}\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|+\log\upsilon_j -H(U_{\mathcal{A}_j},U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c},V_j,Y_j) \quad (6)$$

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}+S_{kj}+T_{kj}\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|+\log\upsilon_j -H(U_{\mathcal{A}_j},U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c},V_j,Y_j) \quad (7)$$

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}+K_j+L_j\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|+H(V_j) -H(U_{\mathcal{A}_j},V_j|U_{\mathcal{A}_j^c},U_{ij}\oplus U_{kj},Y_j) \quad (8)$$

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}+K_j+L_j+S_{ij}+T_{ij}\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|+\log\upsilon_j +H(V_j)-H(U_{\mathcal{A}_j},V_j,U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c},Y_j) \quad (9)$$

$$S_{\mathcal{A}_j}+T_{\mathcal{A}_j}+K_j+L_j+S_{kj}+T_{kj}\leq\sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a|+\log\upsilon_j +H(V_j)-H(U_{\mathcal{A}_j},V_j,U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c},Y_j), \quad (10)$$

for every $\mathcal{A}_j\subseteq\{ji,jk\}$ with distinct indices i,j,k in $\{1,2,3\}$, where $S_{\mathcal{A}_j}\triangleq\sum_{a\in\mathcal{A}_j}S_a,T_{\mathcal{A}_j}\triangleq\sum_{a\in\mathcal{A}_j}T_a,U_{\mathcal{A}_j}=(U_a:a\in\mathcal{A}_j)$ and all the information quantities are evaluated with respect to the state $\Psi^{\underline{U}\,\underline{U}^{\oplus}\,\underline{V}X\underline{Y}}$ characterized at the top of this page. Let α_S denote the convex closure of $\hat{\alpha}_S$. Then $\alpha_S\subseteq\mathscr{C}(\tau)$ is an achievable rate region.

Remark 5. *Inner bound α_S (i) subsumes [19], [20, Thm. 1], (ii) is the CQ analogue of [3, Thm. 7], (iii) does not include a time-sharing Rv and includes the 'don't care' inequalities [21], [22] - (6), (7) with $\mathcal{A}_j=\phi$. Thus, α_S can be enlarged.*

Acknowledgment

We thank Igor Bernard for suggesting an idea that contributed to the development of the proof.

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