Simultaneous Decoding of Classical Coset Codes over 3−User Quantum Interference Channel : New Achievable Rate Regions

Fatma Gouiaa

Arun Padakandla

***Abstract*—We undertake a Shannon-theoretic study of the problem of communicating over a 3−user classical quantum interference channel (3−CQIC) and focus on characterizing inner bounds. Adopting the powerful technique of *tilting*, *smoothing*, and *augmentation* discovered by Sen [1], [2] and combining it with our coset code strategy, we derive a new inner bound to the capacity region of a 3−CQIC. The derived inner bound subsumes all current known bounds and is proven to be strictly larger for identified examples.**

I. Introduction

Info-theoretic study of the CQ capacity of quantum channels can enable us to boost the throughput of current networks by guiding us to design optimal measurements that exploit quantum behavior of optical channels. Additionally, CQ capacity studies [1]–[5] have contributed to the design of insightful [6] qubit communication strategies. These motivate our two-part investigations.

In this article, we consider bit communication over a general 3-CQIC (Fig. 1). We undertake a Shannon-theoretic study and focus on characterizing inner bounds to its CQ capacity region. Our findings yield a new inner bound to its capacity region (Thm. 1) that (i) subsumes the current known largest and (ii) strictly enlarges upon the latter for identified examples. Our companion article presents analogous results for a quantum broadcast channel (QBC).

Let us begin by recalling the current known best coding strategy for a 3−CQIC which is based on Han-Kobayashi's [7] (HK) technique of message splitting via superposition coding. In this strategy, designed in the context of a 2−user interference channel (IC), each transmitter (Tx) splits its message into two parts, which are encoded using two *unstructured IID* codes - a public U−code and a private X−code. In addition to decoding both its parts, each receiver (Rx) decodes into the public U−code of the *other (interfering)* Tx. This enables each Rx to decode and peel off a **univariate** function (component) of the interference.

A generalization of the above coding strategy using *unstructured IID* codes is the current best known coding strategy for a 3−CQIC. To discuss its efficiency, we momentarily ignore quantum non-commutativity and reason as we would for a classical 3−user IC. Specifically, the generalized strategy combines the above HK technique and Marton's [8] precoding (binning) to enable each Rx to decode **univariate** functions (components) of those signals causing it interference. However, note that in contrast to a 2−user IC, each Rx i suffers interference from **two** interfering signals X_j, X_k. The interference it suffers is therefore, in general, some **bivariate** function $V_i = f_i(X_j, X_k)$ of the two interfering signals or a randomization of V_i. Since Rx i needs to only peel interference off, it is appropriate to enable it to *efficiently* decode the outcome V_i of the *relevant bivariate function* as against to decoding univariate components $U_j{=}g_j(X_j)$,$U_k{=}g_k(X_k)$. Is this strategy with *unstructured IID* codes efficient in enabling Rxs to decode bivariate functions?

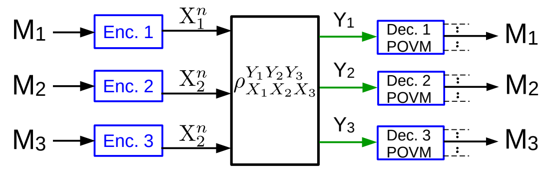

Fig. 1. Three independent messages to be communicated over a 3-CQIC.

Decoding the outcome of bivariate functions efficiently requires the range of the outcomes to be contained (kept small). If X_j, X_k−codes possess no joint structural relationships, the range of applying the bivariate function on the codes explodes. Indeed, while the binary addition of *two cosets* of a *linear code* is contained to be *another coset of the same rate*, *binary addition of unstructured IID random codes explodes*. Building on the early work of Körner and Marton's [9], [10]–[15] have proven that coding strategies based on *jointly* designed coset codes possessing *inter- & intra- algebraic closure* properties coupled with *new encoding decoding* rules are strictly more efficient in most network scenarios involving bivariate function decoding. Our *main goal* here is to go beyond our first demonstration [16] over a 3−CQIC to design a *general* coset code based strategy. Before we discuss the challenges we overcome, let us put forth our main contributions.

Going beyond [16], here, we simultaneously decode into coset and unstructured IID codes and thereby manage all interferences. Analyzing its performance, we derive a new inner bound (Thm. 1) that subsumes the largest achievable (Rem. 3) using unstructured IID codes. Importantly, we identify additive (Ex. 1) and non-additive (Ex. 2) examples, **both non-commutative**, for which the derived inner bound is *analytically* proven to be strictly larger. Proofs are provided in [17]. The derived inner bound can be extended naturally via the standard technique of regularization [18] to derive an inner bound on the bit carrying capacity region of a 3−user quantum broadcast channel.

A general 3-CQIC strategy has to enable interference decoding at all Rxs involving both univariate and bivariate components. This involves multiple codes, and optimality requires Rxs to decode codewords simultaneously (or jointly). Our first challenge is to perform and analyze *simultaneous* *decoding into coset codes*. Exploiting joint algebraic closure of coset codes will entail altering the decoder POVMs, as discussed below, and we will have to simultaneously decode using these new decoder POVMs. Our second challenge relates to achieving inner bounds corresponding to *arbitrary input distributions*. While use of unstructured IID codes lets us choose codewords IID from any target distribution, imposing algebraic closure forces codewords to be *pairwise independent* and *uniformly distributed*. We then have to employ binning [8] to sieve out codewords whose empirical distribution matches the target distribution. Analyzing binning leads us to confront the same challenges encountered in the analysis of a 2−user QBC. For the same reasons discussed in [19], the overcounting trick of [20], [21] does not work here for coset codes.

While simultaneous decoding is straightforward in classical channels, its study posed a major stumbling block since Winter's recognition of the same [5, Sec. VIII]. Following several attempts, Sen [1], [2] overcame this important stumbling block via novel ideas of *tilting, smoothing,* and *augmentation* (TSA). While Sen's TSA enables decoding into specific codes, we are required to *simultaneously decode into 'functions of codebooks'*. Indeed, deriving new inner bounds via coset codes hinges on exploiting the joint algebraic closure property via new decoding POVMs that decode a function of the chosen codewords without nailing down the arguments of the function. The tilting subspaces therefore have to be altered to facilitate this new decoding. By suitably adopting TSA, we perform and analyze *simultaneous* decoding of coset and unstructured codes. Refer to the design of decoder POVMs in the proof of Thm. 1 in Sec. IV-A. To analyze binning of coset codes, we adopt a likelihood encoder that enables us to elegantly separate the analysis of encoding binning and TSA at the decoder. We emphasize that we are able to integrate all these tools in a manner that facilitates analysis.

The study of CQICs has yielded novel approaches [22] tools [23] and techniques [24]. Among related works, Guha, Savov, and Wilde [25] determine the HK region for specific detectors over an optical interference channel. In addition, several studies over CQ multiple access channel (CQMAC) [5] and CQBCs [20], [21] have also contributed to the Shannon-theoretic study of CQICs.

II. Preliminaries and Problem Statement

For $n \in \mathbb{N}$, $[n] \triangleq \{1, \cdots, n\}$. Let $S \subseteq [K]$ be a subset of $[K]$. The complement of S is denoted by $S^c \triangleq [K] \setminus S$. For prime $v \in \mathbb{Z}$, $\mathcal{F}_v$ will denote finite field of size $v \in \mathbb{Z}$ with $\oplus$ denoting field addition in $\mathcal{F}_v$ (i.e. mod$-v$). For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H}), \mathcal{P}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ denote the collection of linear, positive and density operators acting on $\mathcal{H}$ respectively. We let an underline denote an appropriate aggregation of objects. For example, $\underline{\mathcal{X}} \triangleq \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$, $\underline{x} \triangleq (x_1, x_2, x_3) \in \underline{\mathcal{X}}$ and in regards to Hilbert spaces $\mathcal{H}_{Y_i} : i \in [3]$, we let $\mathcal{H}_{\underline{Y}} \triangleq \otimes_{i=1}^{3} \mathcal{H}_{Y_i}$. In any context, when used in conjunction i, j, k will denote distinct indices in [3], hence $\{i, j, k\} = [3]$. We abbreviate probability mass function and conditional typical projector as PMF and C-Typ-Proj respectively.

Consider a (generic) $3-$*CQIC* $(\rho_{\underline{x}} \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : \underline{x} \in \underline{\mathcal{X}}, \kappa_j : j \in [3])$ specified through (i) three finite sets $\mathcal{X}_j : j \in [3]$, (ii) three Hilbert spaces $\mathcal{H}_{Y_j} : j \in [3]$, (iii) a collection $(\rho_{\underline{x}} \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : \underline{x} \in \underline{\mathcal{X}})$ and (iv) three cost functions $\kappa_j : \mathcal{X}_j \rightarrow [0, \infty) : j \in [3]$. The cost function is assumed to be additive, i.e., cost expended by encoder j in preparing the state $\otimes_{t=1}^{n} \rho_{x_{1t}x_{2t}x_{3t}}$ is $\overline{\kappa}_j^n \triangleq \frac{1}{n}\sum_{t=1}^{n} \kappa_j(x_{jt})$. Reliable communication on a $3-$CQIC entails identifying a code.

Defn 1. *A $3-$CQIC code $c = (n, \underline{\mathcal{M}}, \underline{e}, \underline{\mu})$ of block-length n consists of three (i) message index sets $\mathcal{M}_j : j \in [3]$, (ii) encoder maps $e_j : \mathcal{M}_j \rightarrow \mathcal{X}_j^n : j \in [3]$ and (iii) POVMs $\mu_j \triangleq \{\mu_{j,m_j} : \mathcal{H}_j^{\otimes n} \rightarrow \mathcal{H}_j^{\otimes n} : m_j \in \mathcal{M}_j\} : j \in [3]$. Average error probability of the $3-$CQIC code $(n, \underline{\mathcal{M}}, \underline{e}, \underline{\mu})$ is*

$$P(\underline{e}, \underline{\mu}) \triangleq 1 - \frac{1}{|\mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3|} \sum_{\underline{m} \in \underline{\mathcal{M}}} \operatorname{tr}\Big(\mu_{\underline{m}} \rho_{c,\underline{m}}^{\otimes n}\Big),$$

where $\mu_{\underline{m}} \triangleq \otimes_{j=1}^{3} \mu_{j,m_j}$, $\rho_{c,\underline{m}}^{\otimes n} \triangleq \otimes_{t=1}^{n} \rho_{x_{1t}x_{2t}x_{3t}}$ where $(x_{jt} : 1 \leq t \leq n) = x_j^n(m_j) \triangleq e_j(m_j)$ for $j \in [3]$. Average cost per symbol of transmitting message $\underline{m} \in \underline{\mathcal{M}} \in \underline{\tau}(\underline{e}|\underline{m}) \triangleq \big(\overline{\kappa}_j^n(e_j(m_j)) : j \in [3]\big)$ and the average cost per symbol of $3-$CQIC code is $\underline{\tau}(\underline{e}) \triangleq \frac{1}{|\underline{\mathcal{M}}|} \sum_{\underline{m} \in \underline{\mathcal{M}}} \underline{\tau}(\underline{e}|\underline{m})$.

Defn 2. *A rate-cost vector $(R_1, R_2, R_3, \tau_1, \tau_2, \tau_3) \in [0, \infty)^6$ is achievable if there exists a sequence of $3-$CQIC code $(n, \underline{\mathcal{M}}^{(n)}, \underline{e}^{(n)}, \underline{\mu}^{(n)})$ for which $\lim_{n \rightarrow \infty} P(\underline{e}^{(n)}, \underline{\mu}^{(n)}) = 0$,*

$$\lim_{n \rightarrow \infty} n^{-1} \log |\mathcal{M}_j^{(n)}| = R_j, \text{ and } \lim_{n \rightarrow \infty} \underline{\tau}(\underline{e})_j \leq \tau_j : j \in [3].$$

The capacity region $\mathscr{C}$ of the $3-$CQIC is the set of all achievable rate-cost vectors and $\mathscr{C}(\underline{\tau}) \triangleq \{\underline{R} : (\underline{R}, \underline{\tau}) \in \mathscr{C}\}$.

The novelty of our coding scheme is the use of jointly designed coset codes whose structure is explained via the following definition.

Defn 3. *An $(n, k, l, g_I, g_{O/I}, b^n)$ nested coset code over $\mathcal{F}_v$ consists of (i) generator matrices $g_I \in \mathcal{F}_v^{k \times n}$ and $g_{O/I} \in \mathcal{F}_v^{l \times n}$ and (ii) a bias vector $b^n \in \mathcal{F}_v^n$. Let $u^n(a, m) \triangleq a g_I \oplus m g_{O/I} \oplus b^n$, for $(a, m) \in \mathcal{F}_v^k \times \mathcal{F}_v^l$ denote elements in its range space.*

III. Strict suboptimality of unstructured codes for non-commutative Additive & Non-additive 3-CQIC

Our first goal here is to *explain how and why* coset codes yield strictly higher rates over $3-$CQICs. We therefore start with a simple commutative example (Ex. 0) to illustrate ideas. We then build on this example to demonstrate how coset codes can yield higher rates on **non-commutative** additive (Ex. 1) and non-additive (Ex. 2) examples.

Notation for Ex. 0, 1 2 : For $x \in \{0, 1\}$, $\delta \in (0, \frac{1}{2})$, $\sigma_\delta(x) = (1 - \delta)\, |1 - x\rangle\langle 1 - x| + \delta\, |x\rangle\langle x|$. For $\theta \in (0, \frac{\pi}{2})$, $\gamma(x) = |0\rangle\langle 0|\, \mathbb{1}_{\{x=0\}} + |v_\theta\rangle\langle v_\theta|\, \mathbb{1}_{\{x=1\}}$, where $|v_\theta\rangle = [\cos\theta\ \sin\theta]^t$. Let $\mathcal{C}_j(\underline{\tau}) \triangleq \max\{R_j : (R_1, R_2, R_3) \in \mathscr{C}(\underline{\tau})\}$ and $C_j \triangleq \mathcal{C}_j(\infty)$ denote the interference-free capacity of user j with and without input-cost constraints, respectively, for $j \in [3]$.

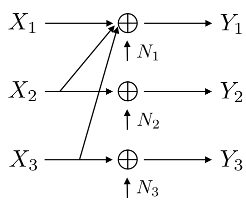

Fig. 2. Equivalent Classical $3-$user IC of Ex. 0

Ex. 0. *Consider a $3-$CQIC with binary inputs and qubit output spaces, i.e. $\mathcal{X}_j = \{0, 1\}$ and $\mathcal{H}_{Y_j} = \mathbb{C}^2$ for $j \in [3]$. The channel is defined as $\rho_{\underline{x}} \triangleq \sigma_{\delta_1}(x_1 \oplus x_2 \oplus x_3) \otimes \sigma_{\delta_2}(x_2) \otimes \sigma_{\delta_3}(x_3)$ for $\underline{x} \in \underline{\mathcal{X}}$. Inputs of users 2 and 3 are not constrained, i.e $\kappa_j(0) = \kappa_j(1) = 0$ for $j = 2, 3$. User 1's input is constrained with respect to a Hamming cost function, i.e $\kappa_1(x) = x$ for $x \in \{0, 1\}$ to an average cost of $\tau \in (0, \frac{1}{2})$ per symbol.*

Throughout, we investigate the achievability of $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ under the condition

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 > C_1 > \mathcal{C}_1 + \max\{\mathcal{C}_2, \mathcal{C}_3\}. \qquad (1)$$

For simplicity let $\delta_2 = \delta_3 = \delta$, (Rem. 2 for general case). Since $\rho_{\underline{x}}$ for $\underline{x} \in \underline{\mathcal{X}}$ are commuting, Ex. 0 can be identified via a $3-$user classical binary IC (Fig.2) with $Y_1 = X_1 \oplus X_2 \oplus X_3 \oplus N_1, Y_k = X_k \oplus N_k$ where $N_1 \sim \text{Ber}(\delta_1)$, $N_k \sim \text{Ber}(\delta)$ for $k = 2, 3$. Inputs X_1 constrained to Hamming cost τ_1 and X_2, X_3 unconstrained. We investigate the achievability of $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ under the condition (1). To answer this, let's fix user k's rate $R_k = \mathcal{C}_k$ for $k = 2, 3$. This forces X_2, X_3 to be independent $\text{Ber}(\frac{1}{2})$ RVs, implying $I(X_1; Y) = 0$. This in turn implies that Rx 1 **has to** decode interference, fully or partly, to achieve *any* positive rate. How does unstructured IID codes enable Rx 1 decode interference?

The unstructured IID coding strategy entails Tx $k \in \{2, 3\}$ split its information into U_k, X_k, Rx 1 decodes U_2, U_3, X_1, while Rx $k \in \{2, 3\}$ decodes U_k, X_k. So long as $H(X_k|U_k) > 0$ for either $k \in \{2, 3\}$ it can be shown [26] that $H(X_2 \oplus X_3|U_2, U_3) > 0$, implying Rx 1 cannot achieve $\mathcal{C}_1$. Indeed, note that it is the interference-free cost constraint capacity of Rx 1 that can't be achieved only if Rx 1 knows full interference. If (1) holds, it can be shown that there exists **no** PMF $p_{X_1}p_{X_2U_2}p_{X_3U_3}$ for which $p_{X_1X_2} = \text{Ber}(\frac{1}{2}) \otimes \text{Ber}(\frac{1}{2})$ **and** $H(X_2|U_2) = H(X_3|U_3) = 0$. Thus (1) precludes unstructured IID codes achieve $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. How can linear codes help under (1)?

User $2, 3$'s channels are simple binary symmetric channel with cross over probability δ (BSC(δ)). Let λ be a linear code of rate $1 - h_b(\delta)$ that achieves capacity of BSC(δ). By employing *cosets λ_2, λ_3 of the same linear code λ* for both users $2, 3$ the collection of all interference patterns - all possible sums of user 2 and 3's codewords - is contained to *another coset $\lambda_2 \oplus \lambda_3$ of the same linear code*. Crucially $\lambda_2 \oplus \lambda_3$ is of the same rate $1 - h_b(\delta)$. Rx 1 just decodes into $\lambda_2 \oplus \lambda_3$ and hunts for the interference $X_2 \oplus X_3$. It can recover the *sum of the codewords* even while being oblivious to the *pair* so long as $\mathcal{C}_1 + \max\{\mathcal{C}_2, \mathcal{C}_3\} < C_1$ holds. In summary, we conclude the following.

Prop 1. *Consider Ex. 0 with τ, δ_1, δ satisfying* (1). *$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is not achievable using any known unstructured IID coding scheme but achievable using coset codes strategy.*

Ex. 1. *Consider a $3-$CQIC binary input sets and qubit output spaces, that is $\mathcal{X}_j = \{0, 1\}$ and $\mathcal{H}_{Y_j} = \mathbb{C}^2$ for $j \in [3]$. The channel is defined as $\rho_{\underline{x}} \triangleq \gamma(x_1 \oplus x_2 \oplus x_3) \otimes \sigma_{\delta_2}(x_2) \otimes \sigma_{\delta_3}(x_3)$, for $\underline{x} \in \underline{\mathcal{X}}$. Inputs of users 2 and 3 are not constrained, i.e $\kappa_j(0) = \kappa_j(1) = 0$ for $j = 2, 3$. User 1's input is constrained with respect to a Hamming cost function, i.e $\kappa_1(x) = x$ for $x \in \{0, 1\}$ to an average cost of $\tau \in (0, \frac{1}{2})$ per symbol.*

The only difference between Ex. 0 and Ex. 1 is the change in System Y_1 which lends the $3-$CQIC to be non-commutative. The same broad line of argument as discussed for Ex. 0, however now incorporating the non-commutative nature via bounds on Holevo information [27], a statement analogous to Prop. 1 can be proven for Ex. 1. The proof of the following is provided in [17].

Prop 2. *Consider the $3-$CQIC in Ex. 1 under condition* (1). *$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is not achievable using any known unstructured coding scheme but is achievable using coset codes strategy.*

We now go one more step. In Exs. 0 and 1, $X_1 \oplus X_2 \oplus X_3$ controlled the Y_1 system received by Rx 1. Since interference $X_2 \oplus X_3$ is linear or additive, one can recognize these gains with coset codes. The question is can we obtain such gains with linear codes even if interference is non-linear. Ex. 2 illustrates that indeed we can obtain gains even in non-additive scenarios.

Ex. 2. *Consider a $3-$CQIC with binary inputs and qubit output spaces, i.e. $\mathcal{X}_j = \{0, 1\}$ and $\mathcal{H}_{Y_j} = \mathbb{C}^2$ for $j \in [3]$. The channel is defined as $\rho_{\underline{x}} \triangleq \gamma\left(x_1 \oplus (x_2 \vee x_3)\right) \otimes \sigma_{\delta_2}(x_2) \otimes \sigma_{\delta_3}(x_3)$ for $\underline{x} \in \underline{\mathcal{X}}$, where $\vee$ denotes the logical OR. User j's inputs is constrained with respect to Hamming cost function, i.e $\kappa_j(x) = x$ for $x \in \{0, 1\}$ to an average cost of $\tau_j \in (0, \frac{1}{2})$ for $j \in [3]$ per symbol.*

While we cannot build codes that are closed wrt to the logical or function, the key idea here is to build coset codes over ternary field and recover the ternary addition. This enables Rx 1 recover $X_2 \vee X_3$ more efficiently that the unstructured coding technique of recovering univariate components of X_2 and X_3. See [17] for details. The following proposition whose proof is provided in [17] establishes this.

Prop 3. *Consider the $3-$CQIC in Ex. 2 under condition* (1). *$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is not achievable using any known unstructured coding scheme but is achievable using coset codes strategy.*

Rem 1. *We note that in the binary field, logical OR is the most non-linear function since all other binary bivariate functions are constants or additive. We are thus able to prove gains for an extreme non-additive example that is also non-commutative. To achieve the rates promised in Prop. 3,*

$$\Gamma^{\underline{U}\underline{U}^{\oplus}\underline{X}\underline{Y}} \triangleq \sum_{\substack{u_{12},u_{13}\\u_{21},u_{23}\\u_{31},u_{32}}} \sum_{\substack{x_1,x_2,x_3\\u_1^{\oplus},u_2^{\oplus},u_3^{\oplus}}} \prod_{j=1}^{3} p_{U_{ji}U_{jk}X_j}(u_{ji},u_{jk},x_j)\mathbb{1}_{\{u_j^{\oplus}=u_{ji}\oplus u_{jk}\}} \left|u_{1*}u_{2*}u_{3*}u_1^{\oplus}u_2^{\oplus}u_3^{\oplus}\underline{x}\right\rangle\!\left\langle u_{1*}u_{2*}u_{3*}u_1^{\oplus}u_2^{\oplus}u_3^{\oplus}\underline{x}\right| \otimes \rho_{x_1x_2x_3}$$

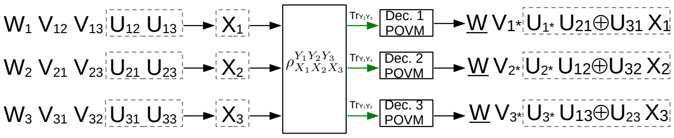

Fig. 3. Depiction of all RVs in the full blown coding strategy. In Sec. IV-A (Step I) only RVs in the gray dashed box are non-trivial, with the rest trivial.

it is crucial to design coset codes over ternary fields and prove achievability of rates corresponding to non-uniform distributions. A general coding strategy must therefore involve arbitrary finite fields and must achieve rates corresponding to arbitrary distributions (see [17]). Our main contribution is to accomplish this via simultaneous decoding of coset codes. We present this general rate region that will decode univariate and bivariate components involving arbitrary distributions in two stages in [17]. The first stage is provided in Sec. IV-A.

Rem 2. *Throughout, we assumed $\delta_2 = \delta_3$, the case of $\delta_2 \neq \delta_3$ can also be easily handled.*

IV. Simultaneous Decoding of Coset Codes

In the following, we describe the coding strategy for the 3-CQIC using coset codes. Tx j splits its message into 6 parts - public W_j, semi-private $V_{j*} \triangleq (V_{ji}, V_{jk})$, 'bivariate' $U_{j*} \triangleq (U_{ji}, U_{jk})$ and private X_j. See Fig. 3. In this coding strategy, support of U_{ji} is finite field $\mathcal{F}_{\upsilon_i}$, while the rest arbitrary finite sets. Rx k decodes all public parts $\underline{W} \triangleq (W_1, W_2, W_3)$, semi-private parts V_{k*}, 'bivariate' $U_k^{\oplus} \triangleq U_{ik} \oplus U_{jk}, U_{k*}$ and finally its private part X_k. We present this final strategy in two steps.

A. Step I : Simultaneous Decoding of Bivariate Components

In this first step, we activate only the 'bivariate' $\underline{U} \triangleq (U_{1*}, U_{2*}, U_{3*})$ and private parts $\underline{X} = (X_1, X_2, X_3)$ are non-trivial, with the rest $\underline{W} = \phi, \underline{V} = (V_{j*} : j \in [3]) = \phi$ trivial. $\underline{X}$ being unstructured IID and $\underline{U}$ coded via coset codes, Thm. 1 possesses all non-trivial elements and meets all objectives.

Theorem 1. *Let $\hat{\alpha}_S \in [0, \infty)^6$ be the set of all rate-cost vectors $(\underline{R}, \underline{\tau})$ for which there exists for all $j \in [3]$, (i) finite fields $\mathcal{U}_{ji} = \mathcal{F}_{\upsilon_i}, \mathcal{U}_{jk} = \mathcal{F}_{\upsilon_k}$, (ii) PMF $p_{\underline{U}\,\underline{X}} = p_{U_{1*}U_{2*}U_{3*}X_1X_2X_3} = \prod_{j=1}^{3} p_{U_{ji}U_{jk}X_j}$ on $\underline{\mathcal{U}} \times \underline{\mathcal{X}}$, (iii) non-negative numbers $S_{ji}, S_{jk}, T_{ji}, T_{jk}, K_j, L_j$ such that $\mathbb{E}\{\kappa_j(X_j)\} \leq \tau_j$, $R_j = T_{ji} + T_{jk} + L_j$ and*

$$S_{A_j} - T_{A_j} + K_j > \sum_{a_j \in A_j} \log|\mathcal{U}_{a_j}| + H(X_j) - H(U_{A_j}, X_j) \quad (2)$$

$$S_{A_j} - T_{A_j} > \sum_{a \in A_j} \log|\mathcal{U}_a| - H(U_{A_j}) \quad (3)$$

$$S_{A_j} < \sum_{a \in A_j} \log|\mathcal{U}_a| - H(U_{A_j}|U_{A_j^c}, U_j^{\oplus}, X_j, Y_j) \quad (4)$$

$$S_{A_j} + S_{ij} < \sum_{a \in A_j} \log|\mathcal{U}_a| + \log q_j - H(U_{A_j}, U_j^{\oplus}|U_{A_j^c}, X_j, Y_j) \quad (5)$$

$$S_{A_j} + S_{kj} < \sum_{a \in A_j} \log|\mathcal{U}_a| + \log q_j - H(U_{A_j}, U_j^{\oplus}|U_{A_j^c}, X_j, Y_j) \quad (6)$$

$$S_{A_j} + K_j + L_j < \sum_{a \in A_j} \log|\mathcal{U}_a| + H(X_j) - H(U_{A_j}, X_j|U_{A_j^c}, U_j^{\oplus}, Y_j), \quad (7)$$

$$S_{A_j} + K_j + L_j + S_{ij} < \sum_{a \in A_j} \log|\mathcal{U}_a| + \log q_j + H(X_j) - H(U_{A_j}, X_j, U_j^{\oplus}|U_{A_j^c}, Y_j) \quad (8)$$

$$S_{A_j} + K_j + L_j + S_{kj} < \sum_{a \in A_j} \log|\mathcal{U}_a| + \log q_j + H(X_j) - H(U_{A_j}, X_j, U_j^{\oplus}|U_{A_j^c}, Y_j), \quad (9)$$

holds for every $j \in [3]$, $A_j \subseteq \{ji, jk\}$, where $U_j^{\oplus} = U_{ij} \oplus U_{kj}$, $S_{A_j} \triangleq \sum_{\nu \in A_j} S_\nu, U_{A_j} = (U_{a_j} : a_j \in A_i)$, where all information quantities are computed wrt state $\Gamma^{\underline{U}\underline{U}^{\oplus}\underline{X}\underline{Y}}$ characterized at the top of this page. Let α_S denote the convex closure of $\hat{\alpha}_S$. Then $\alpha_S \subseteq \mathscr{C}$.

Rem 3. *Inner bound α_S (i) subsumes [16, Thm. 1], (ii) is the CQ analogue of [26, Thm. 3], (iii) does not include a time-sharing RV and includes the 'don't care' inequalities [28], [29] - (5), (6) with $A_j = \phi$. Thus, α_S can be enlarged.*

Discussion of the bounds: We *explain* how (i) each bound arises and (ii) their role in driving error probability down. Fig. 4 depicts codes of user j. Given a message $m_j = (m_{ji}, m_{jk}, m_{jj})$ with three parts, Tx j employs a likelihood encoder. On analyzing the error probability, we obtain 15 source coding bounds [30] that are identical to those one would obtain if Tx j were attempting to find a triple of codewords jointly typical with respect to $p_{U_{ji}U_{jk}X_j}$. (2), (3) are corresponding source coding bounds. To recognize this, through this discussion, we rename the RVs $Z_1 = U_{ji}, Z_2 = U_{jk}, Z_3 = X_j$, PMFs $r_{\underline{Z}} = r_{Z_1Z_2Z_3} = p_{U_{ji}U_{jk}X_j}$, $q_{\underline{Z}} = q_{Z_1Z_2Z_3} = \frac{p_{X_k}}{|\mathcal{U}_{ji}|\cdot|\mathcal{U}_{jk}|}$ and bin sizes $B_1 = S_{ji} - T_{ji}, B_2 = S_{jk} - T_{jk}, B_3 = K_j$. Since $\lambda_{ji}, \lambda_{jk}$ are random coset codes [16] with uniformly distributed codewords and codewords of X_j picked IID wrt $\prod p_{X_j}$, any triple of codewords is distributed with PMF $q_{\underline{Z}}^n$ (Fig. 4). Standard source coding bounds to find one triple to be jointly typical wrt $r_{\underline{Z}}$ among $2^{n(B_1+B_2+B_3)}$ triples are $B_A \overset{(A)}{>} D(r_{Z_A|Z_{A^C}}||q_{Z_A|Z_{A^C}}|r_{Z_{A^C}})$ for every $A \subseteq \{1, 2, 3\}$ where $Z_A = (Z_\iota : \iota \in A)$ and averaged divergence is as defined in [31, Eqn. 2.4]. The 7 bounds in (2), (3) are just the 7 bounds (A) for different choices of $A \subseteq \{1, 2, 3\}$. This

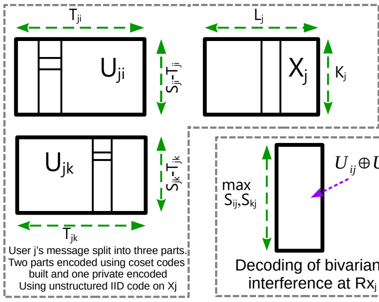

Fig. 4.The box with 3 codes depicts the codes at Tx j. Rx j decodes into these 4 codes.

is verified by direct substitution of $q_{\underline{Z}}, r_{\underline{Z}}$. Imposing (2), (3) guarantees Txs find a triple jointly typical wrt $p_{U_{ji}U_{jk}X_j}$. Now to channel coding bounds (4)-(9). Our decoding analysis is a simple CQMAC decoding at each user. We shall therefore relate (4)-(9) to the bounds one obtains on a CQMAC with 4 Txs. To earlier renaming, we add $Z_4 = U_j^{\oplus} = U_{ij} \oplus U_{kj}$, $Z_5 = Y_j$, $\overline{Z} = Z_1Z_2Z_3Z_4Z_5$,

$$\mathscr{R}^{\overline{Z}} \triangleq \Gamma^{U_{ji}U_{jk}X_jU_j^{\oplus}Y_j} = \mathrm{tr}_{U_{i*}U_{k*}X_iX_kU_i^{\oplus}U_k^{\oplus}Y_iY_k}\{\Gamma^{\underline{U}\,\underline{U}^{\oplus}\,\underline{X}\,\underline{Y}}\},$$

$$\mathscr{Q}^{Z_1Z_2Z_3Z_4} \triangleq \sum_{\substack{(z_1,z_2,z_3,z_4)\\ \in \mathcal{F}_{v_i}\times\mathcal{F}_{v_k}\times\mathcal{X}_j\times\mathcal{F}_{v_j}}} \frac{p_{X_j}(z_3)\,|z_1\ z_2\ z_3\ z_4\rangle\langle z_1\ z_2\ z_3\ z_4|}{|\mathcal{U}_{ji}|\cdot|\mathcal{U}_{jk}|\cdot|\mathcal{U}_{ij}|}$$

$C_1 = S_{ji}, C_2 = S_{jk}, C_3 = K_j + L_j, C_4 = \max\{S_{kj}, S_{ij}\}$. The 7 bounds in (4), (7) are in fact the 7 bounds $C_A < D(\mathscr{R}^{\overline{Z}}||\mathscr{R}^{\overline{Z}_A}\otimes\mathscr{R}^{\overline{Z}_{A^C}}) + D(\mathscr{R}^{\overline{Z}_A}||\mathscr{Q}^{\overline{Z}_A})$ obtained by different choices of $A \subseteq \{1,2,3\}$, and the 16 bounds in (5), (6), (8), (9) are in fact the 8 bounds $C_{A\cup\{4\}} < D(\mathscr{R}^{\overline{Z}}||\mathscr{R}^{\overline{Z}_{A\cup\{4\}}}\otimes\mathscr{R}^{\overline{Z}_{A^C\setminus\{4\}}}) + D(\mathscr{R}^{\overline{Z}_{A\cup\{4\}}}||\mathscr{Q}^{\overline{Z}_{A\cup\{4\}}})$ obtained by different choices of $A \subseteq \{1,2,3\}$ but now replicated twice owing to the max in the definition of C_4. Having identified the bounds, let us interpret the upper bound $D(\mathscr{R}^{\overline{Z}}||\mathscr{R}^{\overline{Z}_A}\otimes\mathscr{R}^{\overline{Z}_{A^C}}) + D(\mathscr{R}^{\overline{Z}_A}||\mathscr{Q}^{\overline{Z}_A})$. Observe that if $\mathscr{Q}^{Z_1Z_2Z_3Z_4} = \mathscr{R}^{Z_1Z_2Z_3Z_4}$, then the upper bound will just have one term which is essentially the bound obtained on a CQMAC. The addition of the $D(\mathscr{R}^{\overline{Z}_A}||\mathscr{Q}^{\overline{Z}_A})$ is due to the fact that codewords of random code are *not* distributed with PMF $p_{U_{ji}U_{jk}X_jU_j^{\oplus}}$, i.e., $\Gamma^{U_{ji}U_{jk}X_j}$, but wrt $\mathscr{Q}^{Z_1Z_2Z_3Z_4}$. This explains (4)-(9). Imposing the same guarantees correct decoding at each Rx.

Proof. In this proof, all subsets $S \subseteq [K]$ are assumed to be non-empty. For brevity, we only provide a characterization of decoding POVM. A full proof can be found in [17]. As discussed in IV-A, each Rx performs CQMAC decoding into 4 codebooks - 3 coset codes and 1 IID code. Specifically, Rx j decodes U_{ji}, U_{jk}, X_j and $U_{ij} \oplus U_{kj}$ using simultaneous decoding (Fig. 4). To reduce clutter and simplify notation, we rename $Z_1 \triangleq U_{ji}, Z_2 \triangleq U_{jk}, Z_3 \triangleq X_j, Z_4 \triangleq U_{ij} \oplus U_{kj}, \underline{Z} \triangleq (Z_1, Z_2, Z_3, Z_4)$, and a PMF $p_{\underline{Z}}(\underline{z}) = \sum_{u_{*j}} p_{U_{j*}X_jU_{*j}}(z_1, z_2, z_3, u_{ij}, u_{kj})\mathbb{1}\{z_4 = u_{ij} \oplus u_{kj}\}$. We, also drop the letter j. We begin by characterizing the effective CQMAC state,

$$\xi^{\underline{Z}Y} \triangleq \sum_{z_1,z_2,z_3,z_4} p_{\underline{Z}}(\underline{z})\,|z_1\ z_2\ z_3\ z_4\rangle\langle z_1\ z_2\ z_3\ z_4| \otimes \xi_{z_1z_2z_3z_4}.$$

Adopting Sen's TSA approach [2], we define $14 = (2^4 - 1) - 1$ tilting maps, one for each error subspace except for the last, where all 4 messages are wrong, which remains untilted. Consider four auxiliary finite sets $\mathcal{D}_i$ for $i \in [4]$ along with corresponding four auxiliary Hilbert spaces $\mathcal{D}_i$ of dimension $|\mathcal{D}_i|$ for each $i \in [4]$.[1] Let

$$(\mathcal{H}_Y^e)^{\otimes n} \triangleq \mathcal{H}_{Y_G}^{\otimes n} \oplus \bigoplus_{S\subseteq[4]\setminus[4]} \left(\mathcal{H}_{Y_G}^{\otimes n} \otimes \mathcal{D}_S^{\otimes n}\right),$$

be the extended space, where $\mathcal{H}_{Y_G}^{\otimes n} = \mathcal{H}_Y^{\otimes n} \otimes \mathbb{C}^2, \mathcal{D}_S^{\otimes n} = \bigotimes_{s\in S}\mathcal{D}_s^{\otimes n}$. Note that $\dim(\mathcal{D}_S) = \prod_{s\in S}\dim(\mathcal{D}_s)$. For $S \subseteq [4]$, let $|d_S\rangle = \otimes_{s\in S}|d_s\rangle$ be a computational basis vector of $\mathcal{D}_S$, $Z_S = (Z_s : s \in S)$, and $0 \le \eta \le \frac{1}{10}$. In the sequel, we let $\boldsymbol{\mathcal{H}_Y} = \mathcal{H}_Y^{\otimes n}$, $\boldsymbol{\mathcal{H}_{Y_G}} = \mathcal{H}_{Y_G}^{\otimes n}$, $\boldsymbol{\mathcal{H}_Y^e} = (\mathcal{H}_Y^{\otimes n})^e$ and $\boldsymbol{\mathcal{D}_S} = \mathcal{D}_S^{\otimes n}$. For $S \subseteq [4] \setminus [4]$, we define the tilting map $\mathcal{T}_{d_S^n,\eta}^S : \boldsymbol{\mathcal{H}_{Y_G}} \to \boldsymbol{\mathcal{H}_{Y_G}} \oplus (\boldsymbol{\mathcal{H}_{Y_G}} \otimes \boldsymbol{\mathcal{D}_S})$ as

$$\mathcal{T}_{d_S^n,\eta}^S(|h\rangle) = \frac{1}{\sqrt{1+\eta^{2|S|}}}\left(|h\rangle + \eta^{|S|}\,|h\rangle \otimes |d_S^n\rangle\right).$$

For $S \subseteq [4]$, consider $G_{\underline{z}^n}^S \triangleq \Pi_{z_{S^c}^n}\Pi_{\underline{z}^n}\Pi_{z_{S^c}^n}$, where $\Pi_{\underline{z}^n}$ and $\Pi_{z_{S^c}^n}$ are C-Typ-Proj wrt the sate $\otimes_{t=1}^n\xi_{z_{1t}z_{2t}z_{3t}z_{4t}}$ and $\otimes_{t=1}^n\left(\xi_{z_{st}:s\in S^c} = \otimes_{t=1}^n\sum_{z_S^n}p_{Z_S|Z_{S^c}}^n(z_S^n|z_{S^c}^n)\xi_{z_{1t}z_{2t}z_{3t}z_{4t}}\right)$ respectively. By Gelfand-Naimark's thm. [27, Thm. 3.7], there exists a projector $\overline{G}_{\underline{z}^n}^S \in \mathcal{P}(\boldsymbol{\mathcal{H}_{Y_G}})$ that yields identical measurement statistic in $\boldsymbol{\mathcal{H}_{Y_G}}$ that $G_{\underline{z}^n}^S$ gives on states in $\boldsymbol{\mathcal{H}_Y}$, for $S \subseteq [4]$. Let $\overline{B}_{\underline{z}^n}^S \triangleq I_{\boldsymbol{\mathcal{H}_{Y_G}}} - \overline{G}_{\underline{z}^n}^S$ be the complement projector. Next, consider

$$\beta_{\underline{z}^n,\underline{d}^n}^S = \mathcal{T}_{d_{S^c}^n,\eta}^{S^c}(\overline{B}_{\underline{z}^n}^S),\ \text{for}\ \ S \ne [4]\ \text{and}\ \beta_{\underline{z}^n,\underline{d}^n}^{[4]} = \overline{B}_{\underline{z}^n}^{[4]},$$

where $\beta_{\underline{z}^n,\underline{d}^n}^S$ represent the tilted projector along direction $d_{S^c}^n$. We define $\beta_{\underline{z}^n,\underline{d}^n}^*$ as the projector in $\boldsymbol{\mathcal{H}_Y^e}$ whose support is the union of the supports of $\beta_{\underline{z}^n,d_S^n}^S$ for all $S \subseteq [4]$. Let $\Pi_{\boldsymbol{\mathcal{H}_{Y_G}}}$ be the orthogonal projector in $\boldsymbol{\mathcal{H}_Y^e}$ onto $\boldsymbol{\mathcal{H}_{Y_G}}$. Let $\underline{m} = (m_1, m_2, m_3, m_4)$, we define the square root measurement $\{\mu_{\underline{m}} : \underline{m} \in \underline{\mathcal{M}}\}$ as

$$\mu_{\underline{m}} \triangleq \left(\sum_{\underline{\hat{m}}}\gamma_{(\underline{z}^n,\underline{d}^n)(\underline{\hat{m}})}^*\right)^{-\frac{1}{2}}\gamma_{(\underline{z}^n,\underline{d}^n)(\underline{m})}^*\left(\sum_{\underline{\hat{m}}}\gamma_{(\underline{z}^n,\underline{d}^n)(\underline{\hat{m}})}^*\right)^{-\frac{1}{2}},$$

$$\mu_{-1} \triangleq I - \sum_{\underline{m}}\mu_{\underline{m}}\ \text{where}$$

$$\gamma_{\underline{z}^n,\underline{d}^n}^* \triangleq \left(I_{\boldsymbol{\mathcal{H}_Y^e}} - \beta_{\underline{z}^n,\underline{d}^n}^*\right)\Pi_{\boldsymbol{\mathcal{H}_{Y_G}}}\left(I_{\boldsymbol{\mathcal{H}_Y^e}} - \beta_{\underline{z}^n,\underline{d}^n}^*\right).$$

We refer the reader to [17] for a complete proof. □

B. Step II: Enlarging α_S via Unstructured IID Codes to α_{US}

Incorporating the HK technique of decoding 'univariate parts' coded via unstructured IID codes, we characterize a inner bound α_{US} that subsumes all current known. This is provided in [17]. A proof of this inner bound is involved, but essentially is a generalization of the proof provided for Thm. 1.

[1] $\mathcal{D}_i$ denotes both the finite set and the corresponding auxiliary Hilbert space. The specific reference will be clear from context.

REFERENCES

[1] P. Sen, "Inner bounds via simultaneous decoding in quantum network information theory," Sādhanā, vol. 46, no. 1, Feb 2021. [Online]. Available: http://dx.doi.org/10.1007/s12046-020-01517-9

[2] ——, "Unions, intersections and a one-shot quantum joint typicality lemma," Sādhanā, vol. 46, no. 1, Mar 2021. [Online]. Available: http://dx.doi.org/10.1007/s12046-020-01555-3

[3] B. Schumacher and M. D. Westmoreland, "Sending classical information via noisy quantum channels," Phys. Rev. A, vol. 56, pp. 131–138, Jul 1997. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevA.56.131

[4] A. Holevo, "The capacity of the quantum channel with general signal states," IEEE Transactions on Information Theory, vol. 44, no. 1, pp. 269–273, 1998.

[5] A. Winter, "The capacity of the quantum multiple-access channel," IEEE Transactions on Information Theory, vol. 47, no. 7, pp. 3059–3065, 2001.

[6] I. Devetak, "The private classical capacity and quantum capacity of a quantum channel," IEEE Transactions on Information Theory, vol. 51, no. 1, pp. 44–55, 2005.

[7] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," IEEE Transactions on Information Theory, vol. 27, no. 1, pp. 49–60, January 1981.

[8] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," IEEE Transactions on Information Theory, vol. 25, no. 3, pp. 306–311, May 1979.

[9] J. Korner and K. Marton, "How to encode the modulo-two sum of binary sources (corresp.)," IEEE Transactions on Information Theory, vol. 25, no. 2, pp. 219–221, 1979.

[10] B. Nazer and M. Gastpar, "Computation over multiple-access channels," IEEE Transactions on Information Theory, vol. 53, no. 10, pp. 3498–3516, 2007.

[11] T. Philosof, R. Zamir, U. Erez, and A. J. Khisti, "Lattice strategies for the dirty multiple access channel," IEEE Transactions on Information Theory, vol. 57, no. 8, pp. 5006–5035, 2011.

[12] T. Philosof and R. Zamir, "On the loss of single-letter characterization: The dirty multiple access channel," IEEE Trans. on Info. Th., vol. 55, pp. 2442–2454, June 2009.

[13] D. Krithivasan and S. Pradhan, "Distributed source coding using abelian group codes: A new achievable rate-distortion region," IEEE Trans. on Info. Th., vol. 57, no. 3, pp. 1495–1519, March 2011.

[14] A. Padakandla and S. S. Pradhan, "Achievable Rate Region for Three User Discrete Broadcast Channel Based on Coset Codes," IEEE Trans. on Information Theory, vol. 64, no. 4, pp. 2267–2297, April 2018.

[15] P. Sen, S. H. Lim, and Y.-H. Kim, "On the optimal achievable rates for linear computation with random homologous codes," IEEE Transactions on Information Theory, vol. 66, no. 10, pp. 6200–6221, 2020.

[16] T. A. Atif, A. Padakandla, and S. S. Pradhan, "Achievable rate-region for 3—User Classical-Quantum Interference Channel using Structured Codes," in 2021 IEEE International Symposium on Information Theory (ISIT), 2021, pp. 760–765.

[17] F. Gouiaa and A. Padakandla, "Achievable rate regions for multi-terminal quantum channels via coset codes," 2025. [Online]. Available: https://arxiv.org/abs/2503.07804

[18] M. M. Wilde, Quantum Information Theory, 2nd ed. Cambridge University Press, 2017.

[19] F. Salek, P. Hayden, and M. Hayashi, "Three-receiver quantum broadcast channels: Classical communication with quantum non-unique decoding," 2024. [Online]. Available: https://arxiv.org/abs/2406.09854

[20] J. Radhakrishnan, P. Sen, and N. Warsi, "One-shot marton inner bound for classical-quantum broadcast channel," IEEE Transactions on Information Theory, vol. 62, no. 5, pp. 2836–2848, 2016.

[21] I. Savov and M. M. Wilde, "Classical codes for quantum broadcast channels," IEEE Transactions on Information Theory, vol. 61, no. 12, pp. 7017–7028, 2015.

[22] P. Sen, "Achieving the han-kobayashi inner bound for the quantum interference channel by sequential decoding," 2011. [Online]. Available: https://arxiv.org/abs/1109.0802

[23] M. M. Wilde, "Sequential decoding of a general classical-quantum channel," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, vol. 469, no. 2157, p. 20130259, 09 2013. [Online]. Available: https://doi.org/10.1098/rspa.2013.0259

[24] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, "Classical communication over a quantum interference channel," IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3670–3691, 2012.

[25] S. Guha, I. Savov, and M. M. Wilde, "The free space optical interference channel," in 2011 IEEE International Symposium on Information Theory Proceedings, 2011, pp. 114–118.

[26] A. Padakandla, A. G. Sahebi, and S. S. Pradhan, "An Achievable Rate Region for the Three-User Interference Channel Based on Coset Codes," IEEE Transactions on Information Theory, vol. 62, no. 3, pp. 1250–1279, March 2016.

[27] A. S. Holevo, Quantum Systems, Channels, Information, 2nd ed. De Gruyter, 2019.

[28] K. Kobayashi and T. Han, "A further consideration on the HK and the CMG regions for the interference channel," San Diego, Feb. 2007.

[29] H.-F. Chong, M. Motani, H. K. Garg, and H. El Gamal, "On the Han-Kobayashi region for the Interference Channel," Information Theory, IEEE Transactions on, vol. 54, no. 7, pp. 3188–3195, 2008.

[30] E. C. Song, P. Cuff, and H. V. Poor, "The likelihood encoder for lossy compression," IEEE Transactions on Information Theory, vol. 62, no. 4, p. 1836–1849, Apr. 2016. [Online]. Available: http://dx.doi.org/10.1109/TIT.2016.2529657

[31] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems, 2nd ed. Cambridge University Press, 2011.