

APPROXIMATING UNIVARIATE FACTORED DISTRIBUTIONS VIA MESSAGE-PASSING ALGORITHMS

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Abstract—Gaussian Mixture Models (GMMs) commonly arise in communication systems, particularly in bilinear joint estimation and detection problems. Although the product of GMMs is still a GMM, as the number of factors increases, the number of components in the resulting product GMM grows exponentially. To obtain a tractable approximation for a univariate factored probability density function (PDF), such as a product of GMMs, we investigate iterative message-passing algorithms. Based on Belief Propagation (BP), we propose a Variable Duplication and Gaussian Belief Propagation (VDBP)-based algorithm. The key idea of VDBP is to construct a multivariate measurement model whose marginal posterior is equal to the given univariate factored PDF. We then apply Gaussian BP (GaBP) to transform the global inference problem into local ones. Expectation propagation (EP) is another branch of message passing algorithms. In addition to converting the global approximation problem into local ones, it features a projection operation that ensures the intermediate functions (messages) belong to a desired family. Due to this projection, EP can be used to approximate the factored PDF directly. However, even if every factor is integrable, the division operation in EP may still cause the algorithm to fail when the mean and variance of a non-integrable belief are required. Therefore, this paper proposes two methods that combine EP with our previously proposed techniques for handling non-integrable beliefs to approximate univariate factored distributions.

I. INTRODUCTION

In bilinear joint channel estimation and data detection problems, the channel likelihood can be characterized as a Gaussian Mixture Model (GMM) [1]. If the data sequence length is greater than one, this GMM becomes the product of multiple simple GMM factors. As the data length increases, the number of simple GMM factors composing the product GMM grows linearly. As a result, the number of components in the product GMM grows exponentially with the number of factors. Although the moments of a single simple GMM with a limited number of components are easy to obtain, the moments of the product GMM in the bilinear estimation and detection problems become intractable as the number of factors increases. In [2]–[4], the authors attempt to avoid GMMs in bilinear joint estimation and detection using various methods. Distributions in the form of a product of GMMs can be generalized as univariate factored distributions. This paper investigates iterative message-passing-based algorithms to approximate the statistics of univariate factored distributions.

A. Prior Works

Given a univariate factored distribution, message-passing algorithms can be used to obtain the marginal distributions, which is equivalent to finding a stationary point of a constrained Bethe Free Energy (BFE) minimization [5].

1) *Belief Propagation*: Belief Propagation (BP) was introduced by Pearl as an approximate inference tool [6]. Within the BFE framework, BP aims to find a stationary point of the BFE under strict marginal consistency constraints. BP is defined on the basis of a factored probability density function (PDF). A corresponding factor graph can be constructed by representing all factors and variables as nodes, and connecting a factor node to a variable node whenever the variable appears as an argument of that factor. By iteratively applying the sum-product rules, BP approximates the marginal distributions via auxiliary functions, called messages [7], [8].

From the constrained BFE optimization perspective, BP seeks a stationary point of the BFE under strict marginal consistency constraints.

This requirement often results in intractable functional forms for the messages. Consequently, even though BP reduces the dimensionality of the inference problem, its computational complexity may still be prohibitive when the messages are intractable—for example, when the factored pdf is a product of GMMs. [9], [10]

Gaussian BP (GaBP) [3], [11] is a variant of BP that projects messages to Gaussian distributions based on the Central Limit Theorem (CLT).

2) *Expectation Propagation*: Expectation Propagation (EP) [12] was proposed to handle intractable messages by introducing a projection step that approximates messages within a chosen family of distributions. In the constrained BFE minimization framework, this corresponds to relaxing the strict marginal consistency constraints to moment constraints [5].

Compared to BP, a major problem with EP is that the factor-level beliefs (i.e., approximated marginals involving the arguments of a factor) may be non-integrable. This prevents EP from proceeding, since the computation of means and variances of such beliefs is required. In [13], the authors investigated this problem and proposed several methods to address it.

B. Main Contribution

This paper investigates both BP- and EP-based approaches for estimating the mean and variance of a univariate factored distribution. BP cannot be directly applied to approximate these statistics when every factor is, for example, a GMM. To address this issue, we propose Variable Duplication Gaussian Belief Propagation (VDBP) to avoid intractable computations. In multivariate inference problems, BP provides marginal distributions for individual variables, as well as joint marginals for variables appearing together in a factor. Leveraging this property, we construct a multivariate linear measurement model whose underlying marginal posterior (or joint PDF with all but one variable marginalized) is equivalent to the given univariate factored distribution up to a constant factor. We then apply GaBP to iteratively approximate the marginal posterior of the multivariate linear measurement model.

As the name suggests, VDBP converts a univariate factored PDF into a multivariate factored PDF by duplicating one variable for each factor, treating the univariate factored distribution as a multivariate joint PDF in which each factor is proportional to the PDF of an independent random variable. The resulting multivariate factored PDF is interpreted as prior information. The measurement channel and measurement outcome are designed such that the measurement likelihood enforces equality among all duplicated variables. Based on the properties of Dirac delta functions, VDBP constructs the measurement matrix as the transpose of the orthogonal complement of the all-ones vector (i.e., each row of the measurement matrix is orthogonal to the all-ones vector). The measurement outcome and noise are both constructed as all-zero vectors. We show analytically that the marginal posterior of the constructed measurement model is equivalent to the original univariate factored PDF.

To overcome the problem of non-integrable beliefs when EP is used to approximate the statistics of univariate factored distributions, we adopt ideas from recent work [13]. We investigated a persistent sequential EP algorithm for estimating the univariate factored PDF, in which updates associated with non-integrable beliefs are skipped. Furthermore, inspired by [13], we propose Analytic Continuation EP (ACEP), in which Gaussian distributions are represented using natural parameters. The technique of analytic continuation allows us to exploit discontinuities in Gaussian functions expressed in natural-parameter form. The key idea of ACEP is to dynamically shrink the projection family in EP to ensure that the belief associated with the next factor is integrable.

Both EP-based algorithms proposed in this paper operate in a sequential manner. We show that the variable-level beliefs produced by both EP based algorithms are guaranteed to be integrable.

II. PROBLEM FORMULATION

Consider a univariate factored PDF with non-negative integrable factors $f_n(\theta)$

$$p(\theta) \propto \prod_{n=1}^N f_n(\theta), \quad (1)$$

where the objective of this paper is to develop a low-complexity method for estimating the statistics of θ . The direct computation of integrals involving $p(\theta)$ becomes intractable when each factor f_n is a mixture model. This paper employs BP- and EP-based algorithms to approximate the statistics of (1). Readers interested in further details are referred to [15] for a more comprehensive introduction to BP and EP.

III. VDBP-BASED APPROXIMATION FOR UNIVARIATE FACTORED PDFS

A. Outline of the VDBP

Inspired by [16], we duplicate the variable θ into N new variables $\{\theta_n\}_{n=1}^N$, one for each factor f_n , to construct a multivariate joint PDF. To describe the equality relationships among these newly constructed variables, we define a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ with $M = N - 1$, such that $\mathbf{A} \cdot \mathbf{1} = \mathbf{0}$. This matrix can be constructed by first generating a full-rank random matrix $\mathbf{B} \in \mathbb{R}^{M \times N}$ and then setting

$$\mathbf{A} = \mathbf{B} - \text{diag} \left(\frac{\mathbf{B} \cdot \mathbf{1}_N}{N} \right) \mathbf{1}_{N-1} \mathbf{1}_N^T. \quad (2)$$

A simple choice for such a matrix is a trimmed Hadamard matrix with the all-ones row removed.

We define the multivariate factored joint PDF as

$$p(\boldsymbol{\theta}) \propto \delta(\mathbf{A}\boldsymbol{\theta}) \prod_n f_n(\theta_n) \propto \prod_m \delta(\mathbf{A}_{m,:} \cdot \boldsymbol{\theta}) \prod_n f_n(\theta_n), \quad (3)$$

where $\mathbf{A}_{m,:}$ denotes the m -th row of the matrix $\mathbf{A}$.

Lemma 1. $\forall \mathbf{A} \in \mathbb{R}^{(N-1) \times N}$ and $\forall n \in \{1, \dots, N\}$, let $\bar{\mathbf{A}}_n$ denote the matrix obtained by removing the n -th column of $\mathbf{A}$. If $\text{rank}(\mathbf{A}) = N - 1$ and $\mathbf{A} \cdot \mathbf{1} = \mathbf{0}$, then $\text{rank}(\bar{\mathbf{A}}_n) = N - 1$.

Proof. See [15] for the proof. $\square$

Theorem 1. $\forall \mathbf{A} \in \mathbb{R}^{M \times N}$, where $M = N - 1$, if $\text{rank}(\mathbf{A}) = M$ and $\mathbf{A} \cdot \mathbf{1} = \mathbf{0}$, then

$$\forall n : \int \delta(\mathbf{A}\boldsymbol{\theta}) \prod_{n' \neq n} f_{n'}(\theta_{n'}) d\theta_{n'} \propto \prod_{n'=1}^N f_{n'}(\theta_n). \quad (4)$$

Proof. See [15] for the proof. $\square$

Algorithm 1 VDBP for GMM Factors

```

1: Initialize  $\forall m, n : \mu_{m,n}^\theta = 0, \tau_{m,n}^\theta = 1$ 
2: repeat
3:    $\forall m : \mu_m^p = \sum_n a_{mn} \mu_{m,n}^\theta$ 
4:    $\forall m : \tau_m^p = \sum_n |a_{mn}|^2 \tau_{m,n}^\theta$ 
5:    $\forall n, m : \mu_{n,m}^p = \mu_m^p - a_{mn} \mu_{m,n}^\theta$ 
6:    $\forall n, m : \tau_{n,m}^p = \tau_m^p - |a_{mn}|^2 \tau_{m,n}^\theta$ 
7:    $\forall n : \tau_n^r = \left( \sum_m \frac{|a_{mn}|^2}{\tau_{n,m}^p + \epsilon_m} \right)^{-1}$ 
8:    $\forall n : \mu_n^r = \tau_n^r \left( \sum_m \frac{a_{mn}^* (y_m - \mu_{n,m}^p)}{\tau_{n,m}^p + \epsilon_m} \right)$ 
9:    $\forall m, n : \tau_{m,n}^r = \left( \tau_n^r - \frac{|a_{mn}|^2}{\tau_{n,m}^p + \epsilon_m} \right)^{-1}$ 
10:   $\forall m, n : \mu_{m,n}^r = \tau_{m,n}^r \left( \frac{\mu_n^r}{\tau_n^r} - \frac{a_{mn}^* (y_m - \mu_{n,m}^p)}{\tau_{n,m}^p + \epsilon_m} \right)$ 
11:   $\forall m, n : \text{Compute normalized } \pi_n(s_n) \text{ via (8)}$ 
12:   $\forall m, n : \text{Compute } \mu_{m,n}^\theta, \tau_{m,n}^\theta \text{ via (9), (10)}$ 
13: until Convergence

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Based on Theorem 1, we can view (1) as a marginalized version of (3), i.e.,

$$\forall i \in \mathbb{N} \cap [1, N] : \prod_n f_n(\theta_i) \propto \int \delta(\mathbf{A}\boldsymbol{\theta}) \prod_n f_n(\theta_n) d\bar{\boldsymbol{\theta}}_i, \quad (5)$$

where $\bar{\boldsymbol{\theta}}_i$ denotes an $(N - 1) \times 1$ vector obtained by removing the i -th entry from $\boldsymbol{\theta}$.

The delta distribution $\delta(\mathbf{A}\boldsymbol{\theta})$ can be interpreted as a noiseless linear measurement model:

$$\mathbf{y} = \mathbf{A}\boldsymbol{\theta} + \mathbf{v}, \quad (6)$$

where $\mathbf{y} = \mathbf{0}$ and $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \text{diag}([\epsilon_1 \dots \epsilon_N]))$, with infinitesimal ϵ_n for all n . The auxiliary noise variance ϵ_n can serve as a regularization term to prevent zero denominators that may occur due to poor initialization. In our simulations, we set $\epsilon_n = 0$. The corresponding factor graph for (3) and (6) can be found in [15]. We then apply GaBP to perform inference on the probabilistic model in (3). The detailed derivation can be found in [15].

B. GMM Factors

Assume that the factors in (1) have the following structure:

$$f_n(\theta_n) \propto \sum_{s_n} p_{s_n} \mathcal{N}(\theta_n | \mu_{s_n}, \tau_{s_n}). \quad (7)$$

The VDBP algorithm for GMM factors [15] is summarized in Algorithm 1, where

$$\pi_n(s_n | \mu_{m,n}^r, \tau_{m,n}^r) \propto p_{s_n} \mathcal{N}(0 | \mu_{s_n} - \mu_{m,n}^r, \tau_{m,n}^r + \tau_{s_n}), \quad (8)$$

and

$$\mu_{m,n}^\theta = \mathbb{E}_{\pi_n(s_n | \mu_{m,n}^r, \tau_{m,n}^r)} [\mu_{m,n}^{x|s}(s_n)], \quad (9)$$

$$\tau_{m,n}^\theta = \mathbb{E}_{\pi_n(s_n | \mu_{m,n}^r, \tau_{m,n}^r)} [\tau_{m,n}^{x|s}(s_n) + \mu_{m,n}^{x|s}(s_n)^2] - (\mu_{m,n}^\theta)^2. \quad (10)$$

IV. PRELIMINARIES FOR EP

In EP, a message does not necessarily correspond to a normalizable function and may even be non-integrable. Let $\mathcal{UN}(\theta | \mu, \tau)$ denote an unnormalized Gaussian (ignoring the $\sqrt{2\pi\tau}$ normalization factor),

$$\mathcal{UN}(\theta | \mu, \tau) = \exp - \frac{(\theta - \mu)^2}{2\tau}. \quad (11)$$

We define a Gaussian distribution in natural-parameter form as

$$\mathcal{M}(\theta | \nu, \xi) = \frac{\sqrt{\xi}}{\sqrt{2\pi}} \exp \left(-\frac{\xi}{2} \theta^2 + \nu \theta - \frac{\nu^2}{2\xi} \right). \quad (12)$$

Similarly, the unnormalized Gaussian in natural-parameter form is defined as

$$\mathcal{UM}(\theta|\nu, \xi) = \exp\left(-\frac{\xi}{2}\theta^2 + \nu\theta - \frac{\nu^2}{2\xi}\right). \quad (13)$$

One can verify that if $\mathcal{UN}(\theta|\mu, \tau) = \mathcal{UM}(\theta|\nu, \xi)$, or if $\mathcal{N}(\theta|\mu, \tau) = \mathcal{M}(\theta|\nu, \xi)$, then

$$\nu = \frac{\mu}{\tau}; \quad \xi = \frac{1}{\tau}. \quad (14)$$

A. Factor Graph of the Given Problem in EP

Now we investigate EP-based algorithms. In contrast to the BP-based VDBP algorithm, the EP-based algorithms apply EP directly to the univariate factored PDF in (1) without the variable duplication process. We define the message from the n -th factor to θ as $\Delta_n^{f \rightarrow \theta}(\theta)$ and the message from θ to the n -th factor as $\Delta_n^{f \leftarrow \theta}(\theta)$. The corresponding factor graph is shown in [15].

V. PERSISTENT EP

In EP, the messages are not required to be integrable. Unlike the Gaussian distribution, the unnormalized Gaussian defined in (11) allows negative values of τ . However, the point $\tau = 0$ is an essential discontinuity of (11). Persistent EP operates under a sequential update order. Specifically, the messages to and from one factor are updated at a time, and the latest messages are immediately used when updating the messages associated with the next factor.

Persistent EP does not actively prevent the formation of non-integrable beliefs. Instead, it handles non-integrable beliefs in a passive manner: if the belief at the factor $f_n(\theta)$ is non-integrable, the update of the message from $f_n(\theta)$ to θ is skipped. The detailed derivation for persistent EP with arbitrary univariate factored PDF can be found in [15].

A. GMM Factors

If the factors in the univariate factored PDF in (1) are GMMs, as defined in (7), we summarize the persistent EP algorithm in Algorithm 2, where we have

$$b_{f_n}(\theta) \propto f_n(\theta) \mathcal{UN}(\theta|\mu_n^{f \leftarrow \theta}, \tau_n^{f \leftarrow \theta}), \quad (15)$$

which is guaranteed to have a finite integral value due to the checking in Algorithm 2 line 8. The corresponding belief mean and variance are

$$\mu_n^{b_f} = \mathbb{E}_{b_{f_n}}[\theta], \quad \tau_n^{b_f} = \text{var}_{b_{f_n}}[\theta]. \quad (16)$$

The update in Algorithm 2 line 10 is

$$\tau_n^{f \rightarrow \theta} = \left(\frac{1}{\tau_n^{b_f}} - \frac{1}{\tau_n^{f \leftarrow \theta}}\right)^{-1}; \quad \mu_n^{f \rightarrow \theta} = \tau_n^{f \rightarrow \theta} \left(\frac{\mu_n^{b_f}}{\tau_n^{b_f}} - \frac{\mu_n^{f \leftarrow \theta}}{\tau_n^{f \leftarrow \theta}}\right). \quad (17)$$

For readers who are interested in implementing persistent EP for GMM-factored PDFs, please refer to [15] for a detailed discussion.

Proposition 1. Assume that $b_\theta(\theta)$ is defined as

$$b_\theta(\theta) \propto \prod_n \mathcal{UN}(\theta|\mu_n^{f \rightarrow \theta}, \tau_n^{f \rightarrow \theta}). \quad (18)$$

If the algorithm in Algorithm 2 is initialized with messages such that $b_\theta(\theta)$ is integrable, then even with an arbitrary univariate factored PDF, the integral of $b_\theta(\theta)$ remains finite throughout all iterations.

Proof. See [15] for the proof. $\square$

After the last iteration, the persistent EP approximates the univariate factored PDF by (18).

Algorithm 2 Persistent EP for GMM Factors

```

1: Initialize  $\forall n : \mu_n^{f \rightarrow \theta} = 0, \tau_n^{f \rightarrow \theta} = 1$ 
2: repeat
3:   [For every  $n \in [1, \dots, N]$ , repeat the following]
4:    $\tau_n^{b_\theta} = \left(\sum_n \frac{1}{\tau_n^{f \rightarrow \theta}}\right)^{-1}$ 
5:    $\mu_n^{b_\theta} = \tau_n^{b_\theta} \left(\sum_n \frac{\mu_n^{f \rightarrow \theta}}{\tau_n^{f \rightarrow \theta}}\right)$ 
6:    $\tau_n^{f \leftarrow \theta} = \frac{\tau_n^{f \rightarrow \theta} \tau_n^{b_\theta}}{\tau_n^{f \rightarrow \theta} - \tau_n^{b_\theta}}$ 
7:    $\mu_n^{f \leftarrow \theta} = \frac{\tau_n^{f \rightarrow \theta} \mu_n^{b_\theta} - \tau_n^{b_\theta} \mu_n^{f \rightarrow \theta}}{\tau_n^{f \rightarrow \theta} - \tau_n^{b_\theta}}$ 
8:   if  $b_{f_n}(\theta)$  in (15) is integrable then
9:     Update  $\mu_n^{b_f}$  and  $\tau_n^{b_f}$  by (15) - (16)
10:    Update  $\tau_n^{f \rightarrow \theta}$  and  $\mu_n^{f \rightarrow \theta}$  by (17)
11:   end if
12: until Convergence

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VI. ANALYTIC CONTINUATION EP

To avoid non-integrable beliefs that prevent the algorithm from progressing, we adopt the technique of analytic continuation proposed in [13]. While persistent EP handles non-integrable beliefs in a passive manner, Analytic Continuation EP (ACEP) actively prevents non-integrable factor-level belief at the subsequent factor.

The key idea of ACEP is to shrink the domain of the projection family, such that the factored-level belief obtained by EP rules at the subsequent factor is integrable.

In ACEP, it is more convenient to represent Gaussian distributions using the natural parameters (ν, ξ) rather than the mean–variance pair (μ, τ) . Detailed derivation for ACEP with both arbitrary factors and GMM factors can be found in [15].

A. GMM Factors

As an illustrative example, consider Gaussian mixture factors of the form given in (7)

$$f_n(\theta) \propto \sum_{s_n} p_{s_n} \mathcal{M}(\theta|\nu_{s_n}, \xi_{s_n}), \quad (19)$$

where

$$\nu_{s_n} = \frac{\mu_{s_n}}{\tau_{s_n}}; \quad \xi_{s_n} = \frac{1}{\tau_{s_n}}. \quad (20)$$

The ACEP algorithm for Gaussian mixture model factors is summarized in Algorithm 3, where the factor-level belief is computed as

$$b_{f_n}(\theta) \propto f_n(\theta) \mathcal{UM}(\theta|\nu_n^{f \leftarrow \theta}, \xi_n^{f \leftarrow \theta}). \quad (21)$$

Its corresponding mean and variance are

$$\mu_n^{b_f} = \mathbb{E}_{b_{f_n}}[\theta]; \quad \tau_n^{b_f} = \text{var}_{b_{f_n}}[\theta]. \quad (22)$$

In Algorithm 3 line 10, since the integrability of a GMM is determined by the signs of its component variances/precisions, we define the threshold for $\xi_n^{f \rightarrow \theta}$ as

$$\begin{aligned} \xi_n^{\text{thres}} &= -\min_{s_k}(\xi_{s_k}) - \prod_{n' \neq n, n' \neq k} \xi_{n'}^{f \rightarrow \theta} \\ &= -\min_{s_k}(\xi_{s_k}) - \xi_n^{f \leftarrow \theta} + \xi_k^{f \rightarrow \theta}. \end{aligned} \quad (23)$$

Let $k = (n \bmod N) + 1$, and b_{f_k} be the next factor-level belief to be computed. From (23), $b_{f_k}(\theta)$ remains integrable if $\xi_n^{f \rightarrow \theta} > \xi_n^{\text{thres}}$.

Algorithm 3 ACEP for GMM Factors

```

1: Initialize  $\forall n : \nu_n^{f \rightarrow \theta} = 0, \xi_n^{f \rightarrow \theta} = 1$ 
2: repeat
3:   [For every  $n \in [1, \dots, N]$ , repeat the following]
4:    $\nu_{b_\theta} = \sum_n \nu_n^{f \rightarrow \theta}$ 
5:    $\xi_{b_\theta} = \sum_n \xi_n^{f \rightarrow \theta}$ 
6:    $\xi_n^{f \leftarrow \theta} = \xi_{b_\theta} - \xi_n^{f \rightarrow \theta}$ 
7:    $\nu_n^{f \leftarrow \theta} = \nu_{b_\theta} - \nu_n^{f \rightarrow \theta}$ 
8:   Update  $\mu_n^{b_f}$  and  $\tau_n^{b_f}$  by (21) and (22)
9:    $k = (n \bmod N) + 1$ 
10:  Update  $\xi_n^{b_\theta}$  by (23)
11:  Update  $\xi_n^{f \rightarrow \theta}$  and  $\nu_n^{f \rightarrow \theta}$  of (24)
12: until Convergence
13: Output  $\mu^{b_\theta}$  and  $\tau^{b_\theta}$  as mean and variance by (25)

```

By solving the KLD projection within a shrunk Gaussian family with ξ_n^{thres} taken into considerations, we have

$$\xi_n^{f \rightarrow \theta} = \begin{cases} \frac{1}{\tau_n^{b_f}} - \xi_n^{f \leftarrow \theta}, & \text{if } \frac{1}{\tau_n^{b_f}} - \xi_n^{f \leftarrow \theta} > \xi_n^{\text{thres}} \\ \xi_n^{\text{thres}}, & \text{otherwise} \end{cases} \quad (24)$$

$$\nu_n^{f \rightarrow \theta} = (\xi_n^{f \rightarrow \theta} + \xi_n^{f \leftarrow \theta}) \mu_n^{b_f} - \nu_n^{f \leftarrow \theta}$$

Proposition 2. Assume that $b_\theta(\theta)$ is defined as

$$b_\theta(\theta) \propto \prod_n \mathcal{UM}(\theta | \nu_n^{f \rightarrow \theta}, \xi_n^{f \rightarrow \theta}). \quad (25)$$

If the algorithm in Algorithm 3 is initialized with messages such that $b_\theta(\theta)$ is integrable, then it remains integrable at every iteration.

Proof. See [15] for the proof. $\square$

VII. SIMULATION RESULTS

To verify the effectiveness of the proposed methods, we generate factored PDFs consisting of $N = 8$ GMM factors in each realization. Each GMM factor contains two Gaussian components with randomly assigned weights, means, and variances. The product of these 8 GMM factors results in another GMM with 2^8 components. To assess the approximation accuracy, we compute the exact mean and variance of the resulting 2^8 -component GMM using brute force, denoted by $\mu_{GMM}^{\hat{\theta}}$ and $\tau_{GMM}^{\hat{\theta}}$ respectively. The normalized squared error (NSE) for a single realization is defined as

$$\begin{aligned} NSE_\mu &= \frac{(\mu^{\hat{\theta}} - \mu_{GMM}^{\hat{\theta}})^2}{(\mu_{GMM}^{\hat{\theta}})^2} \\ NSE_\tau &= \frac{(\tau^{\hat{\theta}} - \tau_{GMM}^{\hat{\theta}})^2}{(\tau_{GMM}^{\hat{\theta}})^2}. \end{aligned} \quad (26)$$

We simulated 10,000 realizations with different sets of GMM factors. The proposed methods are compared with the commonly used Clipping EP [1], [17]. Since checking the integrability of the belief $b_{f_n}(\theta)$ can be computationally expensive, we consider two variants for both proposed EP-based algorithms:

- **Strict approach (STR.):** the algorithms are implemented exactly as proposed.
- **Relaxed approach (RLX.):** for persistent EP, we check the integrability of $\Delta_{f_n}^{f \leftarrow \theta}(\theta)$ instead of $b_{f_n}(\theta)$. For ACEP, we simply set ξ_n^{thres} to zero.

The simulation results for mean estimations are shown in Fig. 1, while the results for variance estimation are shown in Fig. 2. The abbreviation ‘‘PEP’’ denotes persistent EP.

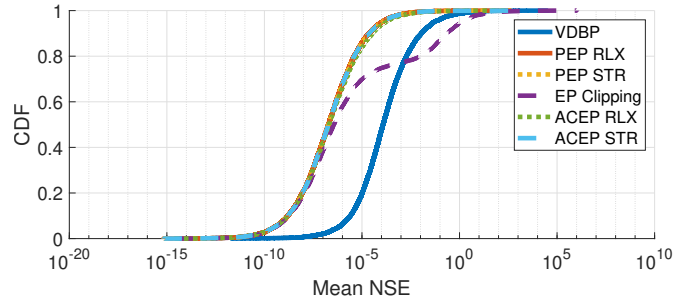


Fig. 1. NSE CDF of mean estimates.

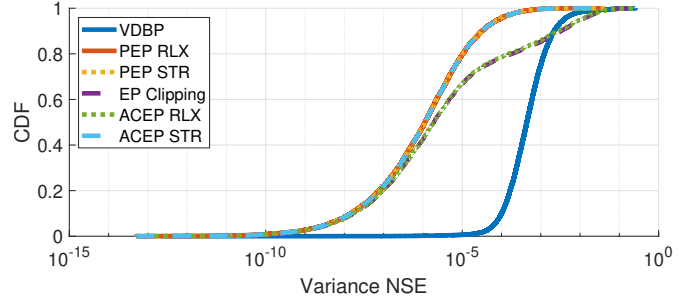


Fig. 2. NSE CDF of variance estimates.

In both figures, the curves corresponding to strict persistent EP, relaxed persistent EP, and strict ACEP coincide and consistently exhibit the best performance. All proposed methods outperform clipping EP in terms of NSE at the 95% CDF.

An interesting observation is that the mean estimate produced by relaxed ACEP is close to the best-performing methods, whereas its variance estimate is close to that of clipping EP. This behavior can be explained by comparing relaxed ACEP with strict ACEP and clipping EP. The variance estimation is closely related to the natural parameter ξ . Both relaxed ACEP and clipping EP disallow non-integrable messages. Therefore, Fig. 2 suggests that the optimal set of messages may include non-integrable messages. By examining the updates of the mean and natural parameter ν in clipping EP and ACEP, we observe that ACEP exploits the limit around the discontinuity $\xi_n^{f \rightarrow \theta} = 0$. In contrast, in clipping EP, when $\tau_n^{f \rightarrow \theta}$ is clipped to $+\infty$, $\mu_n^{f \rightarrow \theta}$ is still treated as a finite value. However, as indicated by ACEP, $\mu_n^{f \rightarrow \theta}$ should instead scale as $O(\tau_n^{f \rightarrow \theta})$.

VIII. CONCLUSIONS

This paper proposes one BP-based algorithm and two EP-based algorithms for estimating the means and variances of univariate factored PDFs.

In the BP-based method, VDBP, we construct a multivariate measurement model whose posterior can be marginalized to the given univariate factored PDF. GaBP is then applied to this multivariate measurement model to approximate the marginal posterior.

EP can be directly applied to estimate the mean and variance of a univariate factored PDF when all factor-level beliefs remain integrable throughout the iterations. To address scenarios involving non-integrable factor-level beliefs, we proposed persistent EP and ACEP.

Persistent EP adopts a passive strategy by skipping message updates around a factor whenever the corresponding belief at the factor is non-integrable. In contrast, ACEP takes an active approach by modifying the projection family to ensure that the belief at the next factor is always integrable.

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