Simultaneous Decoding of Classical Coset Codes over 3−User Quantum Broadcast Channel : New Achievable Rate Regions

Fatma Gouiaa

Arun Padakandla

***Abstract*—We undertake a Shannon theoretic study of the problem of communicating over a 3−user classical quantum broadcast channel (3−CQBC) and focus on characterizing inner bounds. Adopting the technique of *tilting*, *smoothing*, and *augmentation* discovered by Sen [1], [2] and combining it with our coset code strategy, we derive a new inner bound to the capacity region of a 3−CQBC. The derived inner bound subsumes all current known bounds and is proven to be strictly larger for identified examples.**

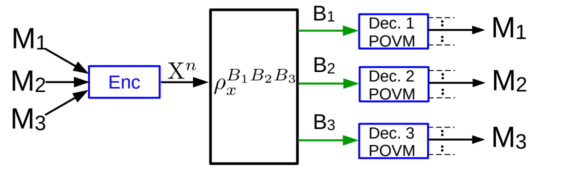

Fig. 1. Three independent messages to be communicated over a 3−CQBC.

I. Introduction

Info-theoretic study of the bit carrying capacity (CQ) of quantum channels (q-channels) guides us in designing measurements that can exploit the quantum behavior of optical channels to enhance throughput over current networks. CQ capacity studies have *also* resulted in novel ideas [3]–[5], resolution of important stumbling blocks [1], [2], all of which have served as building blocks in designing insightful [6] qubit transportation strategies. These motivate our investigations.

In this article, we consider bit communication over a general 3-CQBC (Fig. 1). We undertake a Shannon theoretic study and focus on characterizing inner bounds to its CQ capacity region. Our findings yield a new inner bound to its capacity region that (i) subsumes the current known largest and (ii) strictly enlarges upon the latter for identified examples. Our companion article presents analogous results for a quantum interference channel.

Our findings are based on combining two ideas. The first is based on the use of coset codes endowed with algebraic closure properties. Most works in quantum Shannon theory adopt the conventional approach of designing coding strategies based on unstructured IID codes. The multiple codes employed for communication over a network q-channel are picked independent of each other. Furthermore, codewords of any code are also picked independently. The high probability codes that dominate the average performance possess *only* empirical properties and are bereft of any further structural properties. When communicating over a network q-channel, the codewords from different constituent codes jointly modulate the received quantum state(s). Building on our classical work [7], [8] proves that by employing *nested coset codes* (NCC) and designing a new decoding POVM, we can strictly improve upon all known coding strategies for 3-CQBC that are based on unstructured IID codes. In Secs. II-A, III, we *explain* the phenomenon underlying these findings. Our first idea involves the adoption of NCC for 3−CQBC.

An efficient coding strategy for any CQBC entails receivers (Rxs) decoding into multiple codebooks. Indeed, different codes enable different Rxs to manage their interferences. Starting from [9] several works [10] have pointed to the need for *simultaneous (or joint) decoding* and the sub-optimality of successive decoding in general. While the former is straightforward for classical channels, the skewed orientations/misalignment of joint and conditional typical subspaces lends the analysis of simultaneous decoding for q-channels challenging and requires new approaches. Indeed, following the recognition of this difficulty by Winter [5], different approaches [11], [12] and several attempts [12] culminated in Sen's [2] new ideas of *tilting, smoothing*, and *augmentation* (TSA). Through his works, Sen formulates a robust notion of unions and intersections of subspaces through a process of TSA. Thereby, [1], [2] provides a design for a decoding POVM whose decoding subspaces simultaneously perform the $2^k - 1$ checks that joint decoding into k codebooks entails.

While [2] addresses the challenging one-shot regime and is also important in the broader area of quantum hypothesis testing, it considers only the unstructured IID codebooks. As Ex. 1 illustrates, an efficient coding strategy for a 3−CQBC requires simultaneous decoding into both unstructured IID and NCCs. Our main contribution is the design and analysis of a coding strategy for a general 3−CQBC that adopts ideas of TSA and performs simultaneous decoding into both unstructured IID and NCCs to yield new inner bounds (Thm. 1). A full proof of Thm. 1 is provided in [13]. The derived inner bound can be extended naturally via the standard technique of regularization [14] to derive an inner bound on the bit carrying capacity region of a 3−user quantum broadcast channel.

The adoption of TSA to codes possessing algebraic structure and characterizing new inner bounds involves three difficulties. Firstly, achieving higher rates with coset codes crucially relies on altering the decoding POVM. As illustrated in Ex. 1, the altered POVM must decode into 'functions of codebooks'. Identifying the associated subspaces, tilting maps, and guaranteeing necessary smoothness properties involves aspects not addressed in [1], [2]. We overcome these by resorting back to the appropriately defined conditional typical projectors and tilting them as suggested in [1], [2]. Secondly, in contrast to IID codebooks, the different constituent codes in a random coset code based strategy are statistically dependent. In fact, the coset structure forces codewords of a random coset code to also be only pairwise independent. This lends some of the arguments in the analysis of IID codebooks inapplicable. Thirdly, as recognized in a related work [15], there are technical difficulties in employing the overcounting trick to analyze binning. We overcome this with a likelihood encoder. Combining all these three new elements in a manner that enables analysis is one of our contributions. While this work is a next step to [8], it combines new tools of TSA with coset codes to yield new inner bounds. Sec. IV-A describes tilting of our new decoding POVMs.

An information theoretic study of CQBCs was initiated by Yard et. al. [16] in the context of 2-CQBCs wherein the superposition coding scheme was studied. Furthering this study, Savov and Wilde [17] proved the achievability of Marton's binning [18] for a general 2-CQBC. While these studied the asymptotic regime, Radhakrishnan et. al. [19] proved the achievability of Marton's inner bound [18] in the one-shot regime, which also extended to clear proofs for the former regime considered in [16], [17].

II. Preliminaries and Problem Statement

For $n \in \mathbb{N}$, $[n] \triangleq \{1, \cdots, n\}$. Let $S \subseteq [K]$ be a subset. The complement of S is denoted by $S^c := [K] \setminus S$. For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H}), \mathcal{P}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$ denote the collection of linear, positive, and density operators acting on $\mathcal{H}$ respectively. We let an underline denote an appropriate aggregation of objects. For example, $\underline{\mathcal{V}} \triangleq \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$ for sets, $\mathcal{H}_{\underline{Y}} \triangleq \otimes_{i=1}^3 \mathcal{H}_{Y_i}$ for Hilbert spaces $\mathcal{H}_{Y_i} : i \in [3]$ etc. We abbreviate probability mass function, random variables, and conditional typical projector as PMF, RVs, and C-Typ-Proj respectively.

Consider a (generic) 3-*CQBC* $(\rho_x \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : x \in \mathcal{X}, \kappa)$ specified through (i) a finite set $\mathcal{X}$, (ii) Hilbert spaces $\mathcal{H}_{Y_j} : j \in [3]$, (iii) a collection $(\rho_x \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : x \in \mathcal{X})$ and (iv) a cost function $\kappa : \mathcal{X} \to [0, \infty)$. The cost function is assumed to be additive, i.e., the cost of preparing the state $\otimes_{t=1}^n \rho_{x_t}$ is $\overline{\kappa}^n(x^n) \triangleq \frac{1}{n}\sum_{t=1}^n \kappa(x_t)$. Reliable communication on a 3-CQBC entails identifying a code.

Defn 1. *A* 3*-CQBC code* $c = (n, \underline{\mathcal{M}}, e, \underline{\mu})$ *consists of (i) three message index sets* $\mathcal{M}_j : j \in [3]$*, (ii) an encoder map* $e : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \to \mathcal{X}^n$ *and (iii) POVMs* $\mu_j \triangleq \{\mu_{j,m_j} \in \mathcal{P}(\mathcal{H}_{Y_j}^{\otimes n}) : m_j \in \mathcal{M}_j\} : j \in [3]$*. The average probability of error of the* 3*-CQBC code* $(n, \underline{\mathcal{M}}, e, \underline{\mu})$ *is*

$$P(e, \underline{\mu}) \triangleq 1 - \frac{1}{|\mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3|} \sum_{\underline{m} \in \underline{\mathcal{M}}} \operatorname{tr}\left\{\mu_{\underline{m}} \rho_{c,\underline{m}}^{\otimes n}\right\},$$

where $\mu_{\underline{m}} \triangleq \otimes_{j=1}^3 \mu_{j,m_j}$*,* $\rho_{c,\underline{m}}^{\otimes n} \triangleq \otimes_{t=1}^n \rho_{x_t}$ *and* $(x_t : 1 \le t \le n) = x^n(\underline{m}) \triangleq e(\underline{m})$*. Average cost per symbol of transmitting message* $\underline{m} \in \underline{\mathcal{M}} \in \tau(e|\underline{m}) \triangleq \overline{\kappa}^n(e(\underline{m}))$ *and the average cost per symbol of* 3*-CQBC code is* $\tau(e) \triangleq \frac{1}{|\underline{\mathcal{M}}|}\sum_{\underline{m} \in \underline{\mathcal{M}}} \tau(e|\underline{m})$*.*

Defn 2. *A rate-cost quadruple* $(R_1, R_2, R_3, \tau) \in [0, \infty)^4$ *is achievable if there exists a sequence of* 3*-CQBC codes* $(n, \underline{\mathcal{M}}^{(n)}, e^{(n)}, \underline{\mu}^{(n)})$ *for which* $\lim_{n\to\infty} P(e^{(n)}, \underline{\mu}^{(n)}) = 0$,

$$\lim_{n\to\infty} n^{-1} \log|\mathcal{M}_j^{(n)}| = R_j : j \in [3], \text{ and } \lim_{n\to\infty} \tau(e^{(n)}) \le \tau.$$

The capacity region $\mathscr{C}$ *of a* $3-$*CQBC is the set of all achievable rate-cost vectors and* $\mathscr{C}(\tau) \triangleq \{\underline{R} : (\underline{R}, \tau) \in \mathscr{C}\}$*.*

A. Role of Coset Codes in Quantum Broadcast Channels

How and *why* do coset codes satisfying algebraic closure yield higher rates over a $3-$CQBC? Communication on a BC entails fusing codewords chosen for the different Rxs through a single input. From the perspective of any Rx, a specific aggregation of the codewords chosen for the other Rxs acts as interference. See [20, Fig.2]. The transmitter (Tx) can precode for this interference via Marton's binning [18]. In general, precoding entails a rate loss. In other words, suppose W is the interference seen by Rx 1, then the rate that Rx 1 can achieve by decoding W and peeling it off can be strictly larger than what it can achieve if the Tx precodes for W. This motivates every Rx to decode as large a fraction of the interference that it can and precode only for the minimal residual uncertainty.

In contrast to a BC with 2 Rxs, the interference W on a BC with 3 Rxs is a *bivariate* function of V_2, V_3 - the signals of the other Rxs ([20, Fig.2]). A joint design of the V_2, V_3-codes endowing them with structure can enable Rx 1 decode W efficiently even while being unable to decode V_2 and V_3. This phenomenon is exemplified through Exs. 1.

III. Need for simultaneous decoding of coset codes

We explain the role of coset codes and their need for simultaneous decoding in Ex. 1.

Notation for Ex. 1 : For $b \in \{0, 1\}, \eta \in (0, \frac{1}{2})$, $\sigma_\eta(b) \triangleq (1 - \eta)|1 - b\rangle\langle 1 - b| + \eta|b\rangle\langle b|$, and for $k \in \{2, 3\}$, $\gamma_k(b) = \mathbb{1}_{\{b=0\}}|0\rangle\langle 0| + \mathbb{1}_{\{b=1\}}|v_{\theta_k}\rangle\langle v_{\theta_k}|$, where $|v_{\theta_k}\rangle = [\cos\theta_k \ \sin\theta_k]^t$ and $0 < \theta_2 < \theta_3 < \frac{\pi}{2}$.

Ex. 1. *Let* $\underline{\mathcal{X}} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ *be the input set with* $\mathcal{X}_j = \{0, 1\}$ *for* $j \in [3]$*. For* $\underline{x} = (x_1, x_2, x_3) \in \underline{\mathcal{X}}$*, let* $\rho_{\underline{x}} = \sigma_\epsilon(x_1 \oplus x_2 \oplus x_3) \otimes \beta_2(x_2) \otimes \beta_3(x_3)$*. The symbol* X_1 *is costed via a Hamming cost function i.e., the cost function* $\kappa : \underline{\mathcal{X}} \to \{0, 1\}$ *is given by* $\kappa(\underline{x}) = x_1$*. Investigate* $\mathscr{C}(\tau)$*.*

Commutative Case: $\beta_k(x_k) = \sigma_\delta(x_k)$ for $k = 2, 3$ with $\tau * \epsilon \overset{(i)}{<} \delta$ and $h_b(\delta) \overset{(ii)}{<} \frac{1 + h_b(\tau * \epsilon)}{2}$, where $a * b \triangleq a(1 - b) + (1 - a)b$.

Since $\rho_{\underline{x}} : \underline{x} \in \underline{\mathcal{X}}$ are commuting, the $3-$CQBC for the commuting case can be identified via a 3-user classical BC equipped with input set $\underline{\mathcal{X}}$ and output sets $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3 = \{0, 1\}$. Its input $X = (X_1, X_2, X_3) \in \underline{\mathcal{X}}$ and outputs $Y_j : j \in [3]$ are related via $Y_1 = X_1 \oplus X_2 \oplus X_3 \oplus N_1, Y_j = X_j \oplus N_j$ where N_1, N_2, N_3 are mutually independent Ber(ϵ), Ber(δ), Ber(δ) RVs respectively. We study the question : *What is the maximum achievable rate for Rx* 1 *while Rxs* 2 *and* 3 *are fed at their respective capacities* $C_2 \triangleq C_3 \triangleq 1 - h_b(\delta)$*?*

Choose any $k \in \{2, 3\}$, any $j \in [3] \setminus \{k\}$ and any input PMF $p_X = p_{X_1 X_2 X_3}$. Observe $I(X_j; Y_k) = 0$. Rx 2 and 3 can be pumped information only through X_2 and X_3 respectively. Thus, achieving capacities for Rxs 2, 3 forces $(X_2, X_3) \sim p_{X_2X_3} = \text{Ber}(\frac{1}{2}) \otimes \text{Ber}(\frac{1}{2})$, i.e independent uniform. This implies Rx 1 suffers interference $X_2 \oplus X_3 \sim\text{Ber}(\frac{1}{2})$. With X_1 being Hamming weight constrained to $\tau < \frac{1}{2}$, Tx cannot cancel this interference. Can Rx 1 achieve its interference-free cost constrained capacity $C_1 \triangleq h_b(\tau * \epsilon) - h_b(\epsilon)$?

The relationship $Y_1 = X_1 \oplus [X_2 \oplus X_3] \oplus N_1$ between input X_1 and output Y_1 with $X_2 \oplus X_3 \sim\text{Ber}(\frac{1}{2})$ known only to the Tx is a typical well studied case of *rate loss* [7] in a binary point-to-point channel with Tx side information [10]. These imply that Rx 1 can achieve a rate of C_1 *only if* Rx 1 recovers $X_2 \oplus X_3$ *perfectly*. Can it recover $X_2 \oplus X_3$ *perfectly*?

With **unstructured IID** codes, Rx 1 can recover $X_2 \oplus X_3$ perfectly *only* by recovering X_2 *and* X_3 *perfectly* [7]. However, from inequality $(\boldsymbol{ii})$, we have $1 - h_b(\epsilon) < C_1 + C_2 + C_3$ implying the $X_1 - Y_1$ or the $X - Y_1$ channel **cannot** support Rx 1 decoding X_1, X_2, X_3. Let's consider linear/coset codes. A simple linear coding scheme can achieve the rate triple (C_1, C_2, C_3). In contrast to building independent codebooks for Rxs 2 and 3, choose cosets λ_2, λ_3 of a *common linear code* λ that achieves capacity $1 - h_b(\delta)$ of both $X_2 - Y_2$ and $X_3 - Y_3$ binary symmetric channels. Since the sum of two cosets is another coset of the *same linear* code, the interference patterns seen by Rx 1 are constrained to another coset $\lambda_2 \oplus \lambda_3$ of the same linear code of rate $1 - h_b(\delta)$. Instead of attempting to decode the *pair* of Rx $2, 3$'s codewords, suppose Rx 1 *decodes only the sum of the Rx* $2, 3$ *codewords* within $\lambda_2 \oplus \lambda_3$. Specifically, suppose Rx 1 attempts to jointly decode its codeword and the sum of Rx 2 and 3's codewords, the latter being present in $\lambda_2 \oplus \lambda_3$, then it would be able to achieve a rate of C_1 for itself if $\mathscr{C}_1 = 1 - h_b(\epsilon) > C_1 + \max\{C_2, C_3\} = C_1 + 1 - h_b(\delta)$. The latter inequality is guaranteed since $\tau * \epsilon \overset{(\boldsymbol{i})}{<} \delta$ holds. We conclude this case with the following fact proven in [7].

Fact 1. *For Ex. 1 with inequalities* $(\boldsymbol{i}), (\boldsymbol{ii})$, (C_1, C_2, C_3) *is* ***not*** *achievable via* ***any*** *known unstructured IID coding scheme.* (C_1, C_2, C_3) *is achievable via the above linear code strategy.*

Non-Commuting Case: $\beta_k(x_k) = \gamma_k(x_k)$ for $k = 2, 3$ with $h_b(\tau * \epsilon) + \tilde{h}_b(\cos\theta_2) + \tilde{h}_b(\cos\theta_3) \overset{(\boldsymbol{a})}{>} 1 \overset{(\boldsymbol{b})}{>} h_b(\tau * \epsilon) + \tilde{h}_b(\cos\theta_3)$ where $\tilde{h}_b(x) \triangleq h_b(\frac{1+x}{2})$ for $x \in [0, \frac{1}{2}]$.

We follow the same line of argument as above. For $k = 2, 3$, the capacity of user k is $C_k \triangleq \tilde{h}_b(\cos\theta_k)$ and is achieved *only* by choosing $p_{X_2X_3} = p_{X_2}p_{X_3}$ with each marginal $\text{Ber}(\frac{1}{2})$ [21, Ex.5.6]. This forces Rx 1 to suffer $\text{Ber}(\frac{1}{2})$ interference $X_2 \oplus X_3$. Identical arguments as above revolving around rate loss imply that Rx 1 can achieve a rate of $C_1 \triangleq h_b(\tau * \epsilon) - h_b(\epsilon)$ *only if* Rx 1 recovers $X_2 \oplus X_3$ *perfectly*. The only way unstructured IID codes can enable Rx 1 recover $X_2 \oplus X_3$ *perfectly* is by recovering the *pair* X_2, X_3 *perfectly*. This can be done *only if* the unconstrained capacity $\mathcal{C}_1 = 1 - h_b(\epsilon) \geq C_1 + C_2 + C_3$. However, from $(\boldsymbol{a})$ we have $\mathcal{C}_1 < C_1 + C_2 + C_3$. Unstructured IID codes cannot achieve rate triple (C_1, C_2, C_3).

We design capacity achieving linear codes λ_2, λ_3 for Rxs $2, 3$, respectively, in such a way that λ_2 is a *sub-coset* of λ_3.

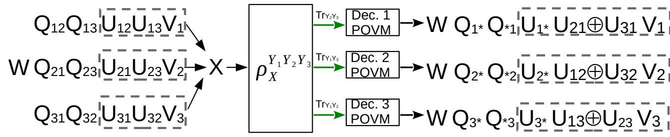

Fig. 2. Depiction of all RVs in the full blown coding strategy. In Sec. IV-A (Step I) only RVs in the grey dashed box are non-trivial, with the rest trivial.

Then, the interference patterns are contained within $\lambda_2 \oplus \lambda_3$ which is now a coset of λ_3. Rx 1 can therefore decode into this collection, which is of rate at most $C_3 = h_b(\tilde{\delta}_3)$. Inequality $(\boldsymbol{b})$ implies the unconstrained capacity $\mathcal{C}_1 = 1 - h_b(\epsilon) \geq C_1 + \max\{C_2, C_3\} = C_1 + C_3$ implying coset codes can achieve rate triple (C_1, C_2, C_3). A proof of this follows from [20, Thm. 1].

Fact 2. *For Ex. 1 with inequalities* $(\boldsymbol{a}), (\boldsymbol{b})$, (C_1, C_2, C_3) *is* ***not*** *achievable via* ***any*** *known unstructured IID coding scheme.* (C_1, C_2, C_3) *is achievable via the above linear code strategy.*

A. Evolving Structure of a General Coding Strategy

Owing to *rate loss*, since (i) decoding is in general more efficient than Tx precoding and (ii) interference on a $3-$CQBC is in general a bivariate function of the interferer signals, Exs. 1 illustrate that by employing jointly designed coset codes and Rxs decoding the bivariate components directly, interference is more efficiently managed. A general coding scheme must facilitate each Rx to efficiently decode *both* univariate and bivariate components of its two interfering signals. Unstructured IID codes and jointly coset codes are efficient at decoding the former and latter components respectively.

Exs. 1 vividly portray the need for enabling Rxs efficiently decode bivariate components of the interference they suffer. A general coding strategy must therefore incorporate codebooks to enable each Rx decoding bivariate functions of the other Rx's signals.

IV. Simultaneous Decoding of Coset Codes

Notation: For prime $\upsilon \in \mathbb{Z}$, $\mathcal{F}_\upsilon$ will denote finite field of size $\upsilon \in \mathbb{Z}$ with $\oplus$ denoting field addition in $\mathcal{F}_\upsilon$ (i.e. mod$-\upsilon$). In any context, when used in conjunction i, j, k will denote distinct indices in [3], hence $\{i, j, k\} = [3]$, and we let $[\![3]\!] \triangleq \{12, 13, 21, 23, 31, 32\}$.

We are led to the following general coding strategy. See Fig. 2. Each Rx j's message is split into six parts $m_j = (m_j^W, m_{ji}^Q, m_{jk}^Q, m_{ji}^U, m_{jk}^U, m_j^V)$. $m_j^W, m_{ji}^Q, m_{jk}^Q$ and m_j^V are encoded using unstructured IID codes built on $\mathcal{W}, \mathcal{Q}_{ji}, \mathcal{Q}_{jk}, \mathcal{V}_j$ respectively. m_{ji}^U and m_{jk}^U are encoded via coset codes built on $\mathcal{U}_{ji} = \mathcal{F}_{\upsilon_i}, \mathcal{U}_{jk} = \mathcal{F}_{\upsilon_k}$ respectively. Rx j decodes $\underline{m}^W, m_{ji}^Q, m_{jk}^Q, m_{ji}^U, m_{jk}^U, m_j^V, m_{ij}^Q, m_{kj}^Q$ and the sum $U_{ij}^n(m_{ij}^U) \oplus U_{kj}^n(m_{kj}^U)$ of the codewords corresponding to m_{ij}^U and m_{kj}^U respectively, where $\underline{m}^W \triangleq (m_1^W, m_2^W, m_3^W)$. We let (i) $Q_{j*} \triangleq (Q_{ji}, Q_{jk})$, (ii) $Q_{*j} \triangleq (Q_{ij}, Q_{kj})$, (iii) $U_{j*} \triangleq (U_{ji}, U_{jk})$, and (iv) $U_j^{\oplus} \triangleq U_{ij} \oplus U_{kj}$ denote RVs encoding (i) semi-private parts m_{ji}^Q, m_{jk}^Q of Rx j, (ii) semi-private parts of Rx i,k decoded by Rx j, (iii) bivariate parts m_{ji}^U, m_{jk}^U of Rx j, and (iv) bivariate component decoded by

$$\Psi^{\underline{U}\underline{U}^{\oplus}\underline{V}X\underline{Y}} \triangleq \sum_{\substack{u_{12},u_{13}\\u_{21},u_{23}\\u_{31},u_{32}}} \sum_{\substack{v_1,v_2,v_3,x\\u_1^{\oplus},u_2^{\oplus},u_3^{\oplus}}} p_{\underline{U}\underline{V}X}(u_{1*},u_{2*},u_{3*},\underline{v},x)\mathbb{1}_{\left\{\substack{u_{ij}\oplus u_{kj}\\=u_j^{\oplus}:j\in[3]}\right\}}\big|u_{1*}\,u_{2*}\,u_{3*}\,u_1^{\oplus}\,u_2^{\oplus}\,u_3^{\oplus}\,\underline{v}\,x\big\rangle\big\langle u_{1*}\,u_{2*}\,u_{3*}\,u_1^{\oplus}\,u_2^{\oplus}\,u_3^{\oplus}\,\underline{v}\,x\big|\otimes\rho_x$$

Rx j respectively. We present the analysis of general strategy in two steps. In this article, we focus on the first step involving only coset codes (Thm. 1). See [13] for a complete proof and a characterization of Step II inner bound.

A. Step I : Simultaneous decoding of Bivariate Components

In this second step, we activate only the 'bivariate' $\underline{U} \triangleq (U_{1*}, U_{2*}, U_{3*})$ and private parts $\underline{V} \triangleq (V_1, V_2, V_3)$, choosing them to be non-trivial, with the rest $\underline{W} = \phi$, $\underline{Q} = (Q_{1*}, Q_{2*}, Q_{3*}) = \phi$ trivial. Since $\underline{V}$ is coded via unstructured IID and $\underline{U}$ coded via coset codes, Thm. 1 addresses all non-trivial elements and meets all objectives.

Theorem 1. *Let $\hat{\alpha}_S \in [0,\infty)^4$ be the set of all rate-cost quadruples $(\underline{R},\tau) \in [0,\infty)^4$ for which there exists (i) finite sets $\mathcal{V}_j : j \in [3]$, (ii) finite fields $\mathcal{U}_{ij} = \mathcal{F}_{v_j}$ for each $ij \in [\![3]\!]$, (iii) a PMF $p_{\underline{U}\underline{V}X} = p_{U_{1*}U_{2*}U_{3*}V_1V_2V_3X}$ on $\underline{\mathcal{U}} \times \underline{\mathcal{V}} \times \mathcal{X}$, (iv) nonnegative numbers $S_{ij}, T_{ij} : ij \in [\![3]\!], K_j, L_j : j \in [3]$, such that $R_1 = T_{12} + T_{13} + L_1, R_2 = T_{21} + T_{23} + L_2, R_3 = T_{31} + T_{32} + L_3$, $\mathbb{E}\{\kappa(X)\} \le \tau$,*

$$S_A + M_B + K_C \overset{\boldsymbol{\alpha}}{>} \Theta(A,B,C), \text{ where} \tag{1}$$

$$\Theta(A,B,C) \triangleq \max_{\substack{(\theta_j:j\in B)\\ \in \prod_{j\in B}\mathcal{F}_{v_j}}} \left\{ \begin{array}{c} \sum_{a\in A}\log|\mathcal{U}_a| + \sum_{j\in B}\log v_j \\ +\sum_{c\in C} H(V_c) \\ -H(U_A, U_{ij}\oplus\theta_j U_{kj} : j\in B, V_C) \end{array} \right\}$$

for all $A \subseteq \{12,13,21,23,31,32\}, B \subseteq \{1,2,3\}, C \subseteq \{1,2,3\}$, that satisfy $A \cap A(B) = \phi$, where $A(B) = \cup_{j\in B}\{ji, jk\}$, $U_A = (U_{jk} : jk \in A)$, $V_C = (V_c : c \in C)$, $S_A = \sum_{jk\in A} S_{jk}$, $M_B \triangleq \sum_{j\in B}\max\{S_{ij} + T_{ij}, S_{kj} + T_{kj}\}$, $K_C = \sum_{c\in C} K_c$, and

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| - H(U_{\mathcal{A}_j}|U_{\mathcal{A}_j^c}, U_{ij}\oplus U_{kj}, V_j, Y_j)$$

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} + S_{ij} + T_{ij} \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| + \log v_j - H(U_{\mathcal{A}_j}, U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c}, V_j, Y_j) \tag{2}$$

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} + S_{kj} + T_{kj} \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| + \log v_j - H(U_{\mathcal{A}_j}, U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c}, V_j, Y_j) \tag{3}$$

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} + K_j + L_j \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| + H(V_j) - H(U_{\mathcal{A}_j}, V_j|U_{\mathcal{A}_j^c}, U_{ij}\oplus U_{kj}, Y_j) \tag{4}$$

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} + K_j + L_j + S_{ij} + T_{ij} \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| + \log v_j + H(V_j) - H(U_{\mathcal{A}_j}, V_j, U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c}, Y_j) \tag{5}$$

$$S_{\mathcal{A}_j} + T_{\mathcal{A}_j} + K_j + L_j + S_{kj} + T_{kj} \le \sum_{a\in\mathcal{A}_j}\log|\mathcal{U}_a| + \log v_j + H(V_j) - H(U_{\mathcal{A}_j}, V_j, U_{ij}\oplus U_{kj}|U_{\mathcal{A}_j^c}, Y_j), \tag{6}$$

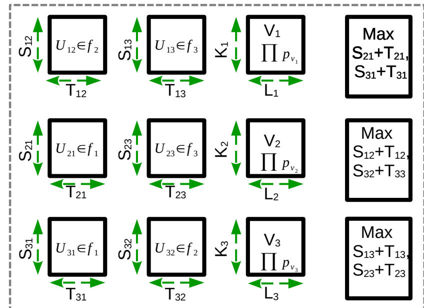

Fig. 3. The 9 codebooks on the left are used by Tx. The 3 rightmost cosets are obtained by adding corresp. cosets. Rx k decodes into 4 codes in row k.

for every $\mathcal{A}_j \subseteq \{ji, jk\}$ with distinct indices i, j, k in $\{1,2,3\}$, where $S_{\mathcal{A}_j} \triangleq \sum_{a\in\mathcal{A}_j} S_a, T_{\mathcal{A}_j} \triangleq \sum_{a\in\mathcal{A}_j} T_a, U_{\mathcal{A}_j} = (U_a : a \in \mathcal{A}_j)$ and all the information quantities are evaluated with respect to the state $\Psi^{\underline{U}\underline{U}^{\oplus}\underline{V}X\underline{Y}}$ characterized at the top of this page. Let α_S denote the convex closure of $\hat{\alpha}_S$. Then $\alpha_S \subseteq \mathscr{C}(\tau)$ is an achievable rate region.

Remark 1. *Inner bound α_S (i) subsumes [8], [20, Thm. 1], (ii) is the CQ analogue of [22, Thm. 7], (iii) does not include a time-sharing RV and includes the 'don't care' inequalities [23], [24] - (2), (3) with $\mathcal{A}_j = \phi$. Thus, α_S can be enlarged.*

Discussion of the bounds: We *explain* how (i) each bound arises and (ii) their role in driving the error probability down. All the source coding bounds are captured through lower bound (1)$\boldsymbol{\alpha}$ by choosing different A, B, C. To understand these bounds, let's begin with a simple example. Suppose we have K codebooks $C_k = (G_k^n(m_k) : m_k \in [2^{nR_k}]) : k \in [K]$ with all of them mutually independent and picked IID in the standard fashion with distribution $G_k^n(m_k) \sim q_{G_k}^n$ for $k \in [K]$. What must the bounds satisfy so that we can find *one* among the $2^{n(R_1+\cdots+R_K)}$ $K-$tuples of codewords to be jointly typical wrt a joint PMF $r_{\underline{G}} = r_{G_1G_2\cdots G_K}$. Denoting $q_{\underline{G}} = \prod_{k=1}^{K} q_{G_k}$, we know from standard info. theory that if $\sum_{l\in S} R_l > D(r_{G_S}|| \otimes_{l\in S} r_{G_l}) + D(\otimes_{l\in S} r_{G_l}|| \otimes_{l\in S} q_{G_l})$ for all $S \subseteq [K]$, then[1] via the standard second moment method [25] we can find the desired $K-$tuple. The bounds in (1)$\boldsymbol{\alpha}$ are just these. We summarize here the detailed explanation provided in [20]. We have 9 codebooks (Fig. 3), however we should also consider the sum of U_{ij} and U_{kj} codebooks, therefore 12 codebooks, hence $K = 12$. $r_{G_1\cdots G_6} = q_{\underline{U}}$ and $q_{U_{ji}} = \frac{1}{|\mathcal{U}_{ji}|}$ is uniform. Indeed, owing to pairwise independence and uniform distribution of codewords in the random coset code [26]. This explains the first term within the max defining $\Theta(\cdots)$. $r_{G_7G_8G_9} = p_{\underline{V}}$ and the $V-$codewords have been picked independently $\prod p_{V_j}$. This explains the third term in the definition of $\Theta(\cdots)$. Owing to the coset structure, it turns out that [7, Bnds. (45), (46)] the U_{ij}-codeword plus θ_j times U_{kj}-codeword must be found in the sum of the U_{ij}, U_{kj}

[1] As is standard, here r_{G_S} is the marginal of $r_{\underline{G}}$ over the RVs indexed in S.

cosets (Fig. 3 rightmost codes). These vectors being uniformly distributed $\sim \frac{1}{v_j}$, we have the second term and the presence of $U_{ji} \oplus \theta_j U_{jk}$ in the entropy term in the defn. of $\Theta(\cdots)$.[2] The channel coding bounds are identical to the ones obtained on a $3-$user CQ interference channel [27].

Proof. In this proof, all subsets $S \subseteq [K]$ are assumed to be non-empty. For brevity, we only provide a characterization of the decoding POVM. See [13] for a full proof. For $j \in [3]$, Rx j's error analysis can be viewed as a standard $4-$user CQ multiple access channel ($4-$CQMAC) error analysis, where the Rx attempts to decode U_{ji}, U_{jk}, V_j and $U_{ij} \oplus U_{kj}$. Henceforth, we use underlined symbols to denote quadruples instead of triples and to simplify notation, we rename $Z_1 \triangleq U_{ji}, Z_2 \triangleq U_{jk}, Z_3 \triangleq V_j, Z_4 \triangleq U_{ij} \oplus U_{kj}$, $p_{\underline{Z}}(\underline{z}) = \sum_{u_{*j}} p_{U_{j*}V_jU_{*j}}(z_1, z_2, z_3, u_{ij}, u_{kj}) \mathbb{1}\{z_4 = u_{ij} \oplus u_{kj}\}$, $\underline{m} \triangleq (m_1, m_2, m_3, m_4) \triangleq (m^U_{ji}, m^U_{jk}, m^V_j, m^U_{ij} \oplus m^U_{kj})$, We also drop the index j, since the decoding analysis is the same for all Rxs. The effective $4-$CQMAC is specified via

$$\xi^{\underline{Z}Y} \triangleq \sum_{z_1,z_2,z_3,z_4} p_{\underline{Z}}(\underline{z}) \left|z_1\ z_2\ z_3\ z_4\right\rangle\!\left\langle z_1\ z_2\ z_3\ z_4\right| \otimes \xi_{z_1z_2z_3z_4}.$$

To build a simultaneous decoding POVM, we employ Sen's TSA technique [2]. We begin by defining $14 = (2^4 - 1) - 1$ tilting maps, one for each error subspace except for the last, where $\underline{m}$ are wrong, which remains untilted. Consider four auxiliary finite sets $\mathcal{D}_i : i \in [4]$ along with corresponding four auxiliary Hilbert spaces $\mathcal{D}_i$ of dimension $|\mathcal{D}_i|$ for each $i \in [4]$.[3] Let

$$(\mathcal{H}^e_Y)^{\otimes n} \triangleq \mathcal{H}^{\otimes n}_{Y_G} \oplus \bigoplus_{S \subsetneq [4]} \left(\mathcal{H}^{\otimes n}_{Y_G} \otimes \mathcal{D}^{\otimes n}_S\right)$$

be the extended space, where $\mathcal{H}^{\otimes n}_{Y_G} = \mathcal{H}^{\otimes n}_Y \otimes \mathbb{C}^2, \mathcal{D}^{\otimes n}_S = \bigotimes_{s \in S} \mathcal{D}^{\otimes n}_s$. Note that $\dim(\mathcal{D}_S) = \prod_{s \in S} \dim(\mathcal{D}_s)$. For $S \subseteq [4]$, let $|d_S\rangle = \otimes_{s \in S} |d_s\rangle$ be a computational basis vector of $\mathcal{D}_S$, $Z_S = (Z_s : s \in S)$, and $0 \le \eta \le \frac{1}{10}$. In the sequel, we let $\boldsymbol{\mathcal{H}_Y} = \mathcal{H}^{\otimes n}_Y$, $\boldsymbol{\mathcal{H}_{Y_G}} = \mathcal{H}^{\otimes n}_{Y_G}$, $\boldsymbol{\mathcal{H}^e_Y} = (\mathcal{H}^{\otimes n}_Y)^e$ and $\boldsymbol{\mathcal{D}_S} = \mathcal{D}^{\otimes n}_S$. For $S \subsetneq [4]$, we define the tilting map $\mathcal{T}^S_{d^n_S,\eta}$ as

$$\mathcal{T}^S_{d^n_S,\eta}(|h\rangle) = \frac{1}{\sqrt{1 + \eta^{2|S|}}} \left(|h\rangle + \eta^{|S|} |h\rangle \otimes |d^n_S\rangle\right).$$

Next, we construct the decoding POVM. For $S \subseteq [4]$, define $G^S_{\underline{z}^n} \triangleq \Pi_{z^n_{S^c}} \Pi_{\underline{z}^n} \Pi_{z^n_{S^c}}$, where $\Pi_{\underline{z}^n}$ and $\Pi_{z^n_{S^c}}$ are C-Typ-Proj wrt the state $\otimes^n_{t=1} \xi_{z_{1t}z_{2t}z_{3t}z_{4t}}$, and $\otimes^n_{t=1} \left(\xi_{z_{st}:s \in S^c} = \otimes^n_{t=1} \sum_{z^n_S} p^n_{Z_S|Z_{S^c}}(z^n_S|z^n_{S^c}) \xi_{z_{1t}z_{2t}z_{3t}z_{4t}}\right)$ respectively. By Gelfand-Naimark's Thm. [21, Thm. 3.7], there exists a projector $\overline{G}^S_{\underline{z}^n} \in \mathcal{P}(\boldsymbol{\mathcal{H}_{Y_G}})$ that yields identical measurement statistic in $\boldsymbol{\mathcal{H}_{Y_G}}$ that $G^S_{\underline{z}^n}$ gives on states in $\boldsymbol{\mathcal{H}_Y}$. Let $\overline{B}^S_{\underline{z}^n} \triangleq I_{\boldsymbol{\mathcal{H}_{Y_G}}} - \overline{G}^S_{\underline{z}^n}$ be the complement projector. Now, we define the tilted projector along direction d^n_S as $\beta^S_{\underline{z}^n,\underline{d}^n} \triangleq \mathcal{T}^{S^c}_{d^n_{S^c},\eta}(\overline{B}^S_{\underline{z}^n})$, for $S \subsetneq [4]$, and the untilted projector as $\beta^{[4]}_{\underline{z}^n,\underline{d}^n} \triangleq \overline{B}^{[4]}_{\underline{z}^n}$. Let $\beta^*_{\underline{z}^n,\underline{d}^n}$ be the projector in $\boldsymbol{\mathcal{H}^e_Y}$ whose support is the union of the supports of $\beta^S_{\underline{z}^n,d^n_S}$ for all $S \subseteq [4]$. Let $\Pi_{\boldsymbol{\mathcal{H}_{Y_G}}}$ be the orthogonal projector in $\boldsymbol{\mathcal{H}^e_Y}$ onto $\boldsymbol{\mathcal{H}_{Y_G}}$. We define $\{\mu_{\underline{m}} : \underline{m}\}$ as

$$\mu_{\underline{m}} \triangleq \left(\sum_{\widehat{\underline{m}}} \gamma^*_{(\underline{z}^n,\underline{d}^n)(\widehat{\underline{m}})}\right)^{-\frac{1}{2}} \gamma^*_{(\underline{z}^n,\underline{d}^n)(\underline{m})} \left(\sum_{\widehat{\underline{m}}} \gamma^*_{(\underline{z}^n,\underline{d}^n)(\widehat{\underline{m}})}\right)^{-\frac{1}{2}},$$

and $\mu_{-1} \triangleq I - \sum_{\underline{m}} \mu_{\underline{m}}$, where

$$\gamma^*_{\underline{z}^n,\underline{d}^n} \triangleq \left(I_{\boldsymbol{\mathcal{H}^e_Y}} - \beta^*_{\underline{z}^n,\underline{d}^n}\right) \Pi_{\boldsymbol{\mathcal{H}_{Y_G}}} \left(I_{\boldsymbol{\mathcal{H}^e_Y}} - \beta^*_{\underline{z}^n,\underline{d}^n}\right).$$

Error Analysis: As is standard, we derive an upper bound on the error probability. We first introduce an alternate 'proxy' state. Specifically, we substitute the state $\xi^{\otimes n}_{\underline{z}^n} \otimes |0\rangle\!\langle 0|$ by a tilted state defined as

$$\theta^{\otimes n}_{\underline{z}^n,\underline{d}^n} \triangleq \mathcal{T}_{\underline{d}^n,\eta}\left(\xi^{\otimes n}_{\underline{z}^n} \otimes |0\rangle\!\langle 0|\right), \text{ where}$$

$$\mathcal{T}_{\underline{d}^n,\eta}(|h\rangle) \triangleq \frac{1}{\sqrt{\Omega(\eta)}} \left(|h\rangle + \sum_{S \subseteq [4], S \neq [4]} \eta^{|S|} |h\rangle \otimes |d^n_S\rangle\right),$$

and $\Omega(\eta) \triangleq 1 + 16\eta^2 + 36\eta^4 + 16\eta^6$. The effect of this substitution on the error probability can be suppressed by ensuring that $\left\|\theta^{\otimes n}_{\underline{z}^n,\underline{d}^n} - \left(\xi^{\otimes n}_{\underline{z}^n} \otimes |0\rangle \langle 0|\right)\right\|_1 \le 12\eta$. The error probability of the $4-$CQMAC is given by

$$P(\underline{e}, \mu) = \frac{1}{|\underline{\mathcal{M}}|} \sum_{\underline{m}} \operatorname{tr}\left\{\left(I - \mu_{\underline{m}}\right) \left(\xi^n_{\underline{Z}}(\underline{m}) \otimes |0\rangle\!\langle 0|\right)\right\} \le t_1 + t_2$$

where $t_1 \triangleq \frac{1}{|\underline{\mathcal{M}}|} \sum_{\underline{m}} \operatorname{tr}\left\{\left(I - \mu_{\underline{m}}\right) \theta_{\underline{Z}^n(\underline{m}),\underline{D}^n(\underline{m})}\right\}$ and

$$t_2 \triangleq \frac{1}{|\underline{\mathcal{M}}|} \sum_{\underline{m}} \left\|\theta_{\underline{Z}^n(\underline{m}),\underline{D}^n(\underline{m})} - \left(\xi_{\underline{Z}^n(\underline{m})} \otimes |0\rangle\!\langle 0|\right)\right\|_1.$$

It is clear that $t_2 \le 12\eta$. To upper bound t_1, we apply Hayashi–Nagaoka inequality [28]. We obtain $t_1 \le t_{1.1} + \sum_{S \subseteq [4]} t_{1.S}$, where

$$t_{1.1} \triangleq \frac{2}{|\underline{\mathcal{M}}|} \sum_{\underline{m}} \operatorname{tr}\left\{(I - \gamma^*_{\underline{Z}^n(\underline{m}),\underline{D}^n(\underline{m})}) \theta_{\underline{Z}^n(\underline{m}),\underline{D}^n(\underline{m})}\right\}, \text{ and}$$

$$t_{1.S} \triangleq \frac{4}{|\underline{\mathcal{M}}|} \sum_{\underline{m}} \sum_{\tilde{m}_S \neq m_S} \operatorname{tr}\left\{\gamma^*_{Z^n_S(\tilde{m}_S), Z^n_{S^c}(m_{S^c}), D^n_S(\tilde{m}_S), D^n_{S^c}(m_{S^c})} \theta_{\underline{Z}^n(\underline{m}),\underline{D}^n(\underline{m})}\right\}.$$

Analogous to the steps in [2], we show that $t_{1.1} \le \frac{336}{\eta^2}(15\epsilon + 30\sqrt{\epsilon}) + 12\eta$. The analysis of t_S differs slightly between $S = [4]$ and $S \neq [4]$. For $S = [4]$, we first establish a required smoothing condition. Since the corresponding projector is not tilted, the remaining steps follow directly by applying it to the averaged original state. For $S \neq [4]$, the analysis is similar for all such S. We again begin by establishing the same smoothing condition. In this case, both the state and the projector are tilted, by extracting the appropriate isometry, as in [2], the analysis reduces to a standard one involving typicality projectors acting on the original state. □

[2]The curious reader may look at the two bounds in [7, Bnds. (72)] with the $\max_{\theta \neq 0}$ which in fact yields [7, Bnds. (45), (46)].

[3]$\mathcal{D}_i$ denotes both the finite set and the corresponding auxiliary Hilbert space. The specific reference will be clear from context.

REFERENCES

[1] P. Sen, "Inner bounds via simultaneous decoding in quantum network information theory," Sādhanā, vol. 46, no. 1, Feb 2021. [Online]. Available: http://dx.doi.org/10.1007/s12046-020-01517-9

[2] ——, "Unions, intersections and a one-shot quantum joint typicality lemma," Sādhanā, vol. 46, no. 1, Mar 2021. [Online]. Available: http://dx.doi.org/10.1007/s12046-020-01555-3

[3] B. Schumacher and M. D. Westmoreland, "Sending classical information via noisy quantum channels," Phys. Rev. A, vol. 56, pp. 131–138, Jul 1997. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevA.56.131

[4] A. Holevo, "The capacity of the quantum channel with general signal states," IEEE Transactions on Information Theory, vol. 44, no. 1, pp. 269–273, 1998.

[5] A. Winter, "The capacity of the quantum multiple-access channel," IEEE Transactions on Information Theory, vol. 47, no. 7, pp. 3059–3065, 2001.

[6] I. Devetak, "The private classical capacity and quantum capacity of a quantum channel," IEEE Transactions on Information Theory, vol. 51, no. 1, pp. 44–55, 2005.

[7] A. Padakandla and S. S. Pradhan, "Achievable Rate Region for Three User Discrete Broadcast Channel Based on Coset Codes," IEEE Trans. on Information Theory, vol. 64, no. 4, pp. 2267–2297, April 2018.

[8] A. Padakandla, "An achievable rate region for 3-user classical-quantum broadcast channels," in 2022 IEEE International Symposium on Information Theory (ISIT), 2022, pp. 2821–2826.

[9] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," IEEE Transactions on Information Theory, vol. 27, no. 1, pp. 49–60, January 1981.

[10] A. El Gamal and Y.-H. Kim, Network Information Theory. Cambridge University Press, 2011.

[11] P. Sen, "Achieving the han-kobayashi inner bound for the quantum interference channel," in 2012 IEEE International Symposium on Information Theory Proceedings, 2012, pp. 736–740.

[12] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, "Classical communication over a quantum interference channel," IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3670–3691, 2012.

[13] F. Gouiaa and A. Padakandla, "Simultaneous decoding of classical coset codes over 3−user quantum broadcast channel: New achievable rate regions," 2025. [Online]. Available: https://arxiv.org/abs/2503.08755

[14] M. M. Wilde, Quantum Information Theory, 2nd ed. Cambridge University Press, 2017.

[15] F. Salek, P. Hayden, and M. Hayashi, "Three-receiver quantum broadcast channels: Classical communication with quantum non-unique decoding," 2024. [Online]. Available: https://arxiv.org/abs/2406.09854

[16] J. Yard, P. Hayden, and I. Devetak, "Quantum broadcast channels," IEEE Transactions on Information Theory, vol. 57, no. 10, pp. 7147–7162, 2011.

[17] I. Savov and M. M. Wilde, "Classical codes for quantum broadcast channels," IEEE Transactions on Information Theory, vol. 61, no. 12, pp. 7017–7028, 2015.

[18] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," IEEE Transactions on Information Theory, vol. 25, no. 3, pp. 306–311, May 1979.

[19] J. Radhakrishnan, P. Sen, and N. Warsi, "One-shot marton inner bound for classical-quantum broadcast channel," IEEE Transactions on Information Theory, vol. 62, no. 5, pp. 2836–2848, 2016.

[20] A. Padakandla, "An achievable rate region for 3−user classical-quantum broadcast channels," 2022. [Online]. Available: https://arxiv.org/abs/2203.00110

[21] A. S. Holevo, Quantum Systems, Channels, Information, 2nd ed. De Gruyter, 2019.

[22] A. Padakandla, A. G. Sahebi, and S. S. Pradhan, "An Achievable Rate Region for the Three-User Interference Channel Based on Coset Codes," IEEE Transactions on Information Theory, vol. 62, no. 3, pp. 1250–1279, March 2016.

[23] K. Kobayashi and T. Han, "A further consideration on the HK and the CMG regions for the interference channel," San Diego, Feb. 2007.

[24] H.-F. Chong, M. Motani, H. K. Garg, and H. El Gamal, "On the Han-Kobayashi region for the Interference Channel," Information Theory, IEEE Transactions on, vol. 54, no. 7, pp. 3188–3195, 2008.

[25] A. El Gamal and E. Van der Meulen, "A proof of Marton's coding theorem for the discrete memoryless broadcast channel," IEEE Trans. Inform. Theory, vol. IT-27, no. 1, pp. 120–122, Jan. 1981.

[26] S. S. Pradhan, A. Padakandla, and F. Shirani, "An algebraic and probabilistic framework for network information theory," Foundations and Trends® in Communications and Information Theory, vol. 18, no. 2, pp. 173–379, 2020. [Online]. Available: http://dx.doi.org/10.1561/0100000083

[27] F. Gouiaa and A. Padakandla, "Achievable rate regions for multi-terminal quantum channels via coset codes," 2025. [Online]. Available: https://arxiv.org/abs/2503.07804

[28] M. Hayashi and H. Nagaoka, "General formulas for capacity of classical-quantum channels," IEEE Transactions on Information Theory, vol. 49, no. 7, pp. 1753–1768, 2003.